Problem Set 6, Math 591, Spring 2009

1. Let $\phi(\lambda)$ be the characteristic function of a random variable. Show that $\phi^2(\lambda)$ and $\mathrm{Re}(\phi(\lambda))$ are also characteristic functions of random variables.

2. Let W_t be a standard Brownian motion. Show that $B_t = \sqrt{a} W_{t/a}$ and $Z_t = tB_{1/t}$ are Brownian motion. (In the latter case you may assume without proof the fact that $\lim_{t\to\infty} W_t/t = 0$ almost surely.)

3. Let W_t be a standard Brownian motion. Show that the joint probability density of $(W_{t_1}, \ldots, W_{t_n})$, for $0 < t_1 < \cdots t_n$, is (with $x_0 := 0$ and $t_0 := 0$)

$$\prod_1^n \frac{e^{-(x_i - x_{i-1})^2/2(t_i - t_{i-1})}}{\sqrt{2\pi(t_i - t_{i-1})}}$$

4. Consider $\{Y_1; q \text{ is rational}, q \in [0,1]\}$ as in Levy's construction of Brownian motion. Let $\{W_t; 0 \leq t \leq 1\}$ be Brownian motion as defined by Levy's construction. Show that $X_t = W_t - tY_1$ is a Gaussian process independent of Y_1 and compute the mean and covariance functions of X_t (i.e., $E[X_t]$ and $\mathrm{Cov}(X_t, X_s)$. Note that $X_0 = 0 = X_1$. X_t is called the Brownian bridge.

5. For a sub-σ-algebra $\mathcal{G}$, prove:

(a) If $0 \leq X_1 \leq X_2 \leq \cdots$, almost surely, and $X = \sup_N X_n$ and $E[X] < \infty$, show that $\lim_{n\to\infty} E[X_n \mid \mathcal{G}] = E[X \mid \mathcal{G}]$ almost surely.

(b) (Conditional Fatou) If X_n are integrable positive random variables, $E[\liminf_{n\to\infty} X_n \;\Big|\; \mathcal{G}] \leq \liminf_{n\to\infty} E[X_n \;\Big|\; \mathcal{G}]$, a.s.

(c) (Conditional dominated convergence) If $|X_n| \leq Y$ for all n and $E[Y] < \infty$, and $X_n \to X$, a.s., then $E[X_n \mid \mathcal{G}] \to E[X \mid \mathcal{G}]$ a.s.

6. (A Bayes rule formula). On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let Z be a positive random variable satisfying $E_P[Z] = 1$, where E_P denotes expectation with respect to $\mathbb{P}$. Define a new probability measure $\tilde{\mathbb{P}}$ by

$$\tilde{\mathbb{P}}(A) = E_P[Z\mathbf{1}_A], \qquad A \in \mathcal{F}.$$

Denote expectations with respect to this probability measure by $E_{\tilde{P}}$. Show that

$$E_{\tilde{P}}[X \mid \mathcal{G}] = \frac{E_P[XZ \mid \mathcal{G}]}{E_P[Z \mid \mathcal{G}]}.$$

7. (a) Let S and T be stopping times with respect to a filtration $\{\mathcal{F}_n\}$. Show that $S \wedge T := \min(S < T)$ and $S \vee T := \max(S, T)$ are stopping times. More generally, show that $\max_n T_n$ and $\min_n T_n$ are stopping times if T_n is a stopping time for every n.

(b) For a $\{\mathcal{F}_n\}$ stopping time T, show that $\mathcal{F}_T := \{A\ ;\ A \cap \{T \le n\} \in \mathcal{F}_n \ \ \forall n < \infty\}$ is a σ-algebra ,

8.. (Polya's Urn Scheme) An urn contains r red and g green balls. At each time n a ball is drawn at random and returned together with c new balls of the same color. Let X_n denote the fraction of green balls in the urn after draw n; $X_0 = g/(r+g)$ is the initial fraction of green balls.

(a) Show that $\{X_n\}$ is a martingale.

(b) Show that $X_\infty = \lim_{n\to\infty} X_n$ exists almost-surely.

9. (A Branching Process) Consider a population of organisms reproducing asexually. Each organism lives for one unit of time and at the end of that time leaves a number of offspring according to a fixed distribution independently of all other organisms. In mathematical notation, let Y_n denote the number of individual in the population at time n. Let $\{Z_i^n\ ;\ i \ge 1, n \ge 1\}$ be i.i.d. random variables taking values in $\{0, 1, 2, \ldots\}$ with mean $m < \infty$. Then $Y_1 = Z_1^1 + \cdots + Z_{Y_0}^1$, (so Z_i^1 is the number of offspring of organism i from the 0^{th} generation); $Y_2 = Z_1^2 + \cdots + Z_{Y_1}^2$, and, in general, $Y_{n+1} = Z_1^{n+1} + \cdots + Z_{Y_n}^{n+1}$. (a) Show that $X_n = m^{-n} Y_n$, $n \ge 0$, is a martingale.

(b) Show that $X_\infty = \lim_{n\to\infty} X_n$ exists almost-surely.

(c) Suppose that there is immigration; that is, $\tilde{Y}_n := Y_n + W_n$ where $W_1, W_2, \ldots$ are i.i.d. with finite mean λ. Show that $\tilde{X}_n = m^{-n}[\tilde{Y}_n - \lambda(1-m^n)/(1-m)]$ is a martingale.

10. Let $\xi_1, \xi_2, \ldots$ be i.i.d. non-negative random variables with mean 1. Define $X_n = \prod_1^n \xi_i$ and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Show that $\{X_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ and that $\lim_{n\to\infty} X_n = 0$. Does there exist a Z such that $X_n = E[Z|\mathcal{F}_n]$?

11. Use the martingale convergence theorem to prove the lemma of Kolmogorov that we used in our discussion of large number laws: if $X_1, X_2, \ldots$ are independent zero mean random variables and $\sum \mathrm{Var}(X_i) < \infty$, then $\sum_1^\infty X_i$ converges almost-surely.

12. Let $\{X_n\}$ be random variables such that; (1) $X_n \to X$ almost surely, and (ii) there exists Y such that $|X_n| \le Y$ almost surely for all n, and $E[Y] < \infty$. Let $\{\mathcal{G}_n\}$ be a filtration, and set $\mathcal{G}_\infty = \sigma(\cup_1^\infty \mathcal{G}_n)$. Show

$$\lim_{n,m\to\infty} E[X_n|\mathcal{G}_m] = E[X|\mathcal{G}_\infty].$$