Problem Set 1, Solutions, 640:591, Spring 2009

1. (a) The inclusion-exclusion formula for $n = 2$, is a consequence of the following two equations, which are direct consequences of finite additivity: $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1 \cap A_2^c) + \mathbb{P}(A_2)$ and $\mathbb{P}(A_1) = \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_2^c)$.

The general case can then be proved by induction.

(b) This can be proved by a combinatorial argument, but it also follows from the inclusion-exclusion formula. To simplify notation, we use AB to denote $A \cap B$.

If $1 \le j_1 < j_2 < \cdots < j_m \le n$, define $1 \le i_1 < \cdots < i_{n-m} \le n$ so that $\{i_1, \ldots, i_{n-m}\} = \{1, \ldots, n\} - \{j_1, \ldots, j_m\}$. If $1 \le s \le n-m$,

$$\sum_{1\le j_1<\cdots<j_m\le n} \sum_{1\le k_1<\cdots<k_s} \mathbb{P}\Big(A_{j_1} \cdots A_{j_m} A_{i_{k_1}} \cdots A_{i_{k_s}}\Big)$$
$$= \binom{m+s}{m} \sum_{1\le \ell_1<\cdots<\ell_{m+s}\le n} \mathbb{P}\Big(A_{\ell_1} \cdots A_{\ell_{m+s}}\Big),$$

because each term of the form $\mathbb{P}\Big(A_{\ell_1} \cdots A_{\ell_{m+s}}\Big)$ appears on the left-hand side $\binom{m+s}{m}$ times, since, for a given $\{\ell_1, \ldots, \ell_{m+s}\}$, there are $\binom{m+s}{m}$ ways to choose the m indices among $\{\ell_1, \ldots, \ell_{m+s}\}$ that constitute $j_1, \ldots, j_m$.

Now observe that

$$\mathbb{P}\Big(A_{j_1} \cdots A_{j_m} A_{i_1}^c \cdots A_{i_{n-m}}^c\Big) = \mathbb{P}\Big(A_{j_1} \cdots A_{j_m}\Big) - \mathbb{P}\Big(\bigcup_{k=1}^{n-m} A_{i_k} A_{j_1} \cdots A_{j_m}\Big).$$

If B_m is the event that exactly m of the A_i's occur, then by applying the inclusion-exclusion formula to the last expression of the previous equation,

$$\begin{aligned}
\mathbb{P}(B_m) &= \sum_{1\le j_1<\cdots<j_m\le n} \mathbb{P}\Big(A_{j_1} \cdots A_{j_m} A_{i_1}^c \cdots A_{i_{n-m}}^c\Big) \\
&= \sum_{1\le j_1<\cdots<j_m\le n} \Big[\mathbb{P}\Big(A_{j_1} \cdots A_{j_m}\Big) \\
&\qquad - \sum_{s=1}^{n-m} (-1)^{s+1} \sum_{1\le k_1<\cdots<k_s\le n-m} \mathbb{P}\Big(A_{j_1} \cdots A_{j_m} A_{i_{k_1}} \cdots A_{i_{k_s}}\Big)\Big] \\
&= \sum_{r=m}^{n} (-1)^{r-m} \binom{r}{m} \sum_{1\le \ell_1<\cdots \ell_r\le n} \mathbb{P}(A_{\ell_1} \cdots A_{\ell_r})
\end{aligned}$$

3. The solution follows from the observation that

$$\{\omega\ ;\ \lim_{n\to\infty} n^{-1} \sum_1^n \omega_i = p\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{\omega;\ \Big| \sum_1^n \omega_i - p \Big| < \frac{1}{k}\}.$$

Each event on the right-hand side is a cylinder set, and hence the event that $\{\omega\ ;\ \lim_{n\to\infty} n^{-1}\sum_1^n \omega_i = p\}$ is in the σ-algebra generated by the cylinder sets.

4. As a preliminary, note that $\Big[\bigcup\limits_1^\infty A_i\Big]\triangle\Big[\bigcup\limits_1^\infty B_i\Big] \subset \bigcup\limits_1^\infty [A_i\triangle B_i]$.

Let $\mathcal{G}$ be the collection of all sets A such that for every $\epsilon > 0$ there exists $A_0 \in \mathcal{R}$ such that $\mathbb{P}(A\triangle A_0) < \epsilon$. We show that $\mathcal{G}$ is a σ-algebra, from which it follows that $\sigma(\mathcal{R}) \subset \mathcal{G}$. Since for any sets A and B, $A\triangle B = A^c\triangle B^c$, $\mathcal{G}$ is closed under the operation of taking complements. Now suppose that $A_1, A_2, \ldots$ are all in $\mathcal{G}$. If $\epsilon > 0$, choose $B_i \in \mathcal{R}$ for every i so that $\mathbb{P}(A_i\triangle B_i) < \epsilon/2^i$. Then, using the first observation and subadditivity of $\mathbb{P}$,

$$\mathbb{P}\Big(\Big[\bigcup_1^\infty A_i\Big]\triangle\Big[\bigcup_1^\infty B_i\Big]\Big) \le \sum_i \mathbb{P}(A_i\triangle B_i) < \epsilon.$$

At the same time, by continuity from above and below,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\Big([\bigcup_1^\infty A_i]\triangle[\bigcup_1^n B_i]\Big) &= \lim_{n\to\infty}\Big\{\mathbb{P}\Big([\bigcup_1^\infty A_i]\cap[\bigcap_1^n B_i^c]\Big) + \mathbb{P}\Big([\bigcup_1^\infty A_i]^c\cap[\bigcup_1^n B_i]\Big)\Big\} \\
&= \mathbb{P}\Big([\bigcup_1^\infty A_i]\cap[\bigcap_1^\infty B_i^c]\Big) + \mathbb{P}\Big([\bigcup_1^\infty A_i]^c\cap[\bigcup_1^\infty B_i]\Big) \\
&= \mathbb{P}\Big([\bigcup_1^\infty A_i]\triangle[\bigcup_1^\infty B_i]\Big)
\end{aligned}$$

Therefore, there is an n such that $\mathbb{P}\Big([\bigcup_1^\infty A_i]\triangle[\bigcup_1^n B_i]\Big) < \epsilon$, and $\bigcup_1^n B_i$ is in $\mathcal{R}$. As $\epsilon > 0$ was arbitrary, this proves that $\mathcal{G}$ is closed under countable unions.