

Problem Set 2, Solutions, 640:591, Spring 2009

1. (a) Let \mathcal{F} be the σ -algebra of subsets of ℓ_2 generated by the cylinder sets. Let \mathcal{B} denote the Borel sets of ℓ_2 . To show that $\mathcal{B} \subset \mathcal{F}$ it suffices to show that every open ball is in \mathcal{F} . (This fact uses the separability of ℓ_2 ; there is a countable dense subset of ℓ_2 , namely all sequences with only a finite number of non-zero terms, all of which are rational. Therefore any open set in ℓ_2 is a countable union of open balls.) Since if $A \in \mathcal{F}$, so is any translate $A + \eta$, it suffices to show any open ball centered at the origin is in \mathcal{F} . But

$$\left\{ \omega \in \ell_2 \mid \sum_i \omega_i^2 < r \right\} = \bigcup_{k \geq 1} \bigcap_{n \geq 1} \left\{ \omega \mid \sqrt{\sum_1^n \omega_i^2} < r - \frac{1}{k} \right\},$$

and the right-hand side is certainly a set in \mathcal{F} .

For any n , subsets of the form $U \times \ell_2$, where U is an open ball in \mathbb{R}^n , generate the σ -algebra of subsets of the form $B \times \ell_2$, where B is a Borel subset of \mathbb{R}^n . Therefore, such sets, where n can be any positive integer, generate \mathcal{F} . However if U is open in \mathbb{R}^n , $U \times \ell_2$ is an open subset of ℓ_2 . It follows that $\mathcal{F} \subset \mathcal{B}$. (A perhaps more elegant expression of this argument is the following. \mathcal{F} is the smallest σ -algebra making each projection map $\pi_n : \ell^2 \rightarrow \mathbb{R}^n$ defined by $\pi_n(\omega) = (\omega_1, \dots, \omega_n)$ measurable as a map into $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. But π_n is continuous and continuous maps are certainly Borel measurable. Hence $\mathcal{F} \subset \mathcal{B}$.)

(b) The proof that ν is finitely additive is the same as for \mathbb{R}^∞ .

Assume that ν has a countably additive extension μ . We will show that this leads to a contradiction. Because P is continuous from above and it is assumed that $\mathbb{P}(\{0\}) < 1$, there exists an $a > 0$ such that $P((-a, a)) < 1$. Let

$$A_n = \bigcap_{m=n}^{\infty} \{\omega \in \ell_2 \mid |\omega_m| \geq a\}.$$

Observe that because $\bigcap_{m=n}^N \{\omega \mid |\omega_i| < a\}$ is a cylinder set,

$$\mu\left(\bigcap_{m=n}^N \{\omega \mid |\omega_i| < a\}\right) = \nu\left(\bigcap_{m=n}^N \{\omega \mid |\omega_i| < a\}\right) = P^{N-n+1}((-a, a)).$$

Then $\mu(A_n) = 1 - \mu(A_n^c) = 1 - \lim_{N \rightarrow \infty} \mu\left(\bigcap_{m=n}^N \{\omega \mid |\omega_i| < a\}\right) = 1 - \lim_{N \rightarrow \infty} P^{N-m+1}((-a, a)) = 1$,

for all n . However, $\{A_n\}$ is a decreasing sequence and $\bigcap_n A_n = \emptyset$, because, there is no $\omega \in \ell_2$ that contains an infinite subsequence of terms greater than a in absolute value. This contradicts the continuity from above of μ .

2. (a) By the assumption that μ and ν are σ -finite, there is a countable partition $\{A_n\}$ of Ω into measurable sets such that $\mu(A_n) < \infty$ for all n , and a countable partition $\{B_n\}$ of Ω into measurable sets such that $\nu(A_n) < \infty$ for all n . Thus the family $\{A_n \cap B_m\}$ constitutes a partition such that the μ measure and the ν measure of each set in the partition are both finite. For convenience, call this partition $\{C_n\}$. To prove the result, it suffices to prove that $\mu_n(A) \leq \nu_n(A)$ for each n , where $\mu_n(A) := \mu(A \cap C_n)$ and $\nu_n(A) := \nu(A \cap C_n)$. This means that it is enough to prove the result for bounded measures.

Therefore, suppose in addition that μ and ν are bounded. Then $\mathcal{M} := \{A \in \mathcal{F}; \mu(A) \leq \nu(A)\}$ is a monotone class, by continuity from above and below of bounded measures. Thus, if $\mathcal{F}_0 \subset \mathcal{M}$ where \mathcal{F}_0 is an algebra generating \mathcal{F} , the monotone class theorem implies that $\mathcal{F} = \mathcal{M}$.

(b) The set of intervals of the form $(-\infty, x]$, where $x \in R$, is a π -system that generates the σ -algebra of Borel sets. For any non-decreasing, right-continuous function $F : R \rightarrow R$, there is a unique measure m_F on the Borel subsets of R satisfying $m_F((-\infty, x]) = F(x)$. Let

$$F_1(x) = \begin{cases} x/2 & \text{if } 0 < x < 2; \\ 0 & \text{if } x \leq 0; \\ 1 & \text{if } x \geq 2. \end{cases}$$

Let

$$F_2(x) = \begin{cases} x - 1 & \text{if } 1 < x < 2; \\ 0 & \text{if } x \leq 1; \\ 1 & \text{if } x \geq 2. \end{cases}$$

Let μ_1 and μ_2 be the associated Borel measures. Then $\mu_1((-\infty, x]) = F_1(x) \geq F_2(x) = \mu_2((-\infty, x])$, for all x . But if $1 \leq a < b \leq 2$, $\mu_1((a, b]) = F_1(b) - F_1(a) = (b - a)/2 < b - a = F_2(b) - F_2(a) = \mu_2((a, b])$.

3. Let \mathcal{C} be closed under finite intersections. Let \mathcal{D} be the smallest λ -system containing \mathcal{C} . We shall prove first that \mathcal{D} is an algebra. Since any λ -system is closed under the operation of taking complements, it suffices to show that \mathcal{D} is closed under finite intersections. To this end, if A is a set in \mathcal{D} , define $\mathcal{D}_A = \{B \in \mathcal{C}; B \cap A \in \mathcal{D}\}$. Then \mathcal{D} is a λ -system. This is easy to check. Now suppose $A \in \mathcal{C}$. Then, because \mathcal{C} is closed under finite intersections, $\mathcal{C} \subset \mathcal{D}_A$. Therefore, since \mathcal{D} is the smallest λ -system containing \mathcal{C} , it follows that $\mathcal{D} \subset \mathcal{D}_A$. This proves that $A \cap B \in \mathcal{D}$ whenever $A \in \mathcal{C}$ and $B \in \mathcal{D}$. But since this is true, if $B \in \mathcal{D}$, then $\mathcal{C} \subset \mathcal{D}_B$. Hence $\mathcal{D} \subset \mathcal{D}_B$. This proves that $B \cap C \in \mathcal{D}$ whenever $B, C \in \mathcal{D}$, which is what we wanted to prove.

Next we observe that a λ -system is a monotone class. Indeed, because it is closed under countable increasing unions and under complementation, it is closed under

countable decreasing intersections. Since \mathcal{D} is an algebra, the monotone class theorem implies that \mathcal{D} is in fact a σ -algebra. Hence $\sigma(\mathcal{C}) \subset \mathcal{D}$. But as a σ -algebra is λ -system, $\mathcal{D} \subset \sigma(\mathcal{C})$. Hence $\sigma(\mathcal{C}) = \mathcal{D}$.

4. Suppose that A_3 is independent of A_1 , A_3 is independent of A_2 but A_1, A_2, A_3 are not mutually independent. then if $\mathcal{G}_1 = \{A_1, A_2\}$ and $\mathcal{G}_2 = \{A_3\}$, \mathcal{G}_1 and \mathcal{G}_2 are independent, but $\sigma(\mathcal{G}_1)$ and \mathcal{G}_2 are not because, A_3 and $A_1 \cap A_2$ are not independent. For an example, let Ω be $\{0, 1\}^2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ with every singleton equally likely. This models two independent tosses of a fair coin. Let $A_1 = \{(1, 0), (1, 1)\}$ (heads on the first toss), $A_2 = \{(0, 1), (1, 1)\}$ (heads on the second toss), and $A_3 = \{(1, 0), (0, 1)\}$ (exactly one head in two tosses). Then $A_1 \cap A_2 \cap A_3 = \emptyset$, so the events are not mutually independent. But it is easy to check that A_3 is independent of A_1 and A_3 is independent of A_2 .

5. (a) This type of example could have also been used in the problem 4. Consider 3 coin independent tosses of a fair coin. For $i \neq j$, let B_{ij} be the event that both toss i and toss j are heads or neither are. Then $\mathbb{P}(B_{ij}) = 1/2$. We claim that B_{12} , B_{13} , and B_{23} are pairwise independent. By symmetry, it suffices to show B_{12} and B_{13} are independent. But $B_{12} \cap B_{13}$ occurs if and only if all three tosses are heads or all are tails, and this has probability $1/4 = \mathbb{P}(B_{12})\mathbb{P}(B_{13})$. However, $B_{12} \cap B_{13} \cap B_{23}$ also occurs if and only if all three tosses are heads or all are tails, so the probability of this intersection is also $1/4$, which is not the product of the probabilities of the individual events, this product being $1/8$. Hence the three events are not mutually independent.

(b) Consider $n+1$ independent tosses of a fair coin. For $i = 1, \dots, n$, let U_i be the event that tosses i and $i+1$ have the same result, both heads or both tails. Let U_{n+1} be the event that toss 1 and toss $n+1$ have the same result. For each i , $\mathbb{P}(U_i) = 1/2$. The intersection $\bigcap_1^{n+1} U_i$ is the event that all the tosses are all the same. It has probability $1/2^n$; as this is not equal to $\prod_1^{n+1} \mathbb{P}(U_i) = 1/2^{n+1}$, the $n+1$ events are not mutually independent. However, it is not hard to see that the members of any subset of n or fewer are independent.