Solutions, Problem Set 3, Math 591, Spring 2009

1. Using Fubini's theorem,

$$E[|X|^p] = \int_\Omega \int_0^{|X|} (p-1)x^{p-1}\, dx\, d\mathbb{P} = \int_0^\infty (p-1)x^{p-1} \int_\Omega \mathbf{1}_{|X|\geq x}\, d\mathbb{P} = \int_0^\infty px^{p-1} \mathbb{P}(|X| \geq x)\, \mathrm{d}x.$$

2. By the argument proving Markov's inequality,

$$x^p \mathbb{P}(|X| \geq x) \leq E\left[\mathbf{1}_{|X|\geq x}|X|^p\right].$$

If $E[|X|^p] < \infty$, the right-hand side converges to 0 as $x \to \infty$, by the dominated convergence theorem.

If $\lim_{x\to\infty} x^p \mathbb{P}(|X| \geq x) = 0$, the function $x \to x^{p-1-\epsilon}\mathbb{P}(|X| \geq x)$ is integrable over $[0, \infty)$, if $\epsilon > 0$. Hence

$$E[|X|^{p-\epsilon}] = \int_0^\infty (p-1-\epsilon)x^{p-1-\epsilon}\mathbb{P}(|X| \geq x)\, dx < \infty.$$

4. Let $\{X_n\}$ be increasing and suppose that $X_n \xrightarrow{P} X$. Since $\{X_n\}$ is increasing, $Y = \lim_{n\to\infty} X_n$ exists (possibly infinite) everywhere. Since there is a subsequence of X_{n_k} that converges almost surely to X, it follows that $X = Y$ almost surely.

A direct proof is also not hard. Since $\liminf \mathbf{1}_{\{|X_n - X| > \epsilon/2\}} \geq \mathbf{1}_{\{|Y-X|>\epsilon\}}$, Fatou's Lemma implies

$$\mathbb{P}(|Y - X| > \epsilon) \leq \liminf \mathbb{P}(|X_n - X| > \epsilon/2) = 0, \quad \text{for all } \epsilon > 0.$$

Letting $\epsilon \downarrow 0$ proves $\mathbb{P}(Y = X) = 1$.

5. (a) Let s be a permutation of $\{1, \ldots, n\}$. Let $Z_i = X_{s(i)}$, $1 \leq i \leq n$. Then the joint distribution $Z_1, \ldots, Z_n$ is the same as that of $X_1, \ldots, X_n$. Let π be the random permutation that orders $X_1, \ldots, X_n$ in increasing order. Then the event $\{\pi = s\}$ is the same as the event that $\{Z_1 < Z_2 < \cdots < Z_n\}$. Thus $\mathbb{P}(\pi = s) = \mathbb{P}(X_1 < \cdots < X_n) = \mathbb{P}(\pi = \mathbf{i})$, where $\mathbf{i}$ is the identity permutation. This does not depend on s and so all permutations are equally likely.

As $Y_n = k$ if and only if $\pi(n) = k$ and there are $(n-1)!$ permutations of $\{1, \ldots, n\}$ with $\pi(n) = k$ and $n!$ permutations of $\{1, \ldots, n\}$ overall,

$$\mathbb{P}(Y_n = k) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Similarly, the event $\{Y_1 = k_1, \ldots, Y_n = k_n\}$, if it is an event that can occur with positive probability, determines a unique π with $X_{\pi(1)} < \ldots < X_{\pi(n)}$, and conversely π determines $Y_1, \ldots, Y_n$. So

$$\mathbb{P}(Y_1 = k_1, \ldots, Y_n = k_n) = \frac{1}{n!} = \prod_{j=1}^{n} \frac{1}{j} = \prod_{j=1}^{n} \mathbb{P}(Y_j = k_j).$$

This establishes that $Y_1, \ldots, Y_n$ are independent for any n. Hence $Y_1, Y_2, \ldots$ are independent.

(b) A record occurs at n at $Y_n = n$. We have $\sum_{j=0}^{\infty} \mathbb{P}(Y_n = n) = \sum_{j=0}^{\infty} \frac{1}{n} = \infty$. By the Borel-Cantelli Lemma (second half), $\mathbb{P}(Y_n = n \ \text{ infinitely often}) = 1$.

6. $\mathrm{Cov}(X, Y)$ is bilinear in its arguments X and Y. Since X' and X'' are independent, $\mathrm{Cov}(f(X'), g(X'')) = 0$ and $\mathrm{Cov}(f(X''), g(X')) = 0$. It follows that $\mathrm{Cov}(f(X') - f(X''), g(X') - g(X'')) = 2\mathrm{Cov}(f(X), g(X))$. Since f and g are increasing $(f(X') - f(X''))(g(X') - g(X'')) \geq 0$ everywhere. Thus,

$$\mathrm{Cov}(f(X), g(X)) = (1/2)E\left[(f(X') - f(X''))(g(X') - g(X''))\right] \geq 0.$$

7. This problem can be done using Kolmogorov's inequality—see class notes on the law of large numbers. However, the conclusion is true without assuming independence, only that the random variables are identically distributed. Here is the argument. Observe that

$$\mathbb{P}\left(\frac{\max\{|X_i|;\, i \leq n\}}{\sqrt{n}} > \epsilon\right) = \mathbb{P}\left(\bigcup_{1}^{n}\{|X_i| > \epsilon\sqrt{n}\}\right) \leq \sum_{1}^{n} \mathbb{P}\left(|X_i| > \epsilon\sqrt{n}\right).$$

Because the random variables are identically distributed,

$$\mathbb{P}\left(\frac{\max\{|X_i|;\, i \leq n\}}{\sqrt{n}} > \epsilon\right) \leq n\mathbb{P}\left(|X_1| > \epsilon\sqrt{n}\right) \leq n\frac{E\left[|X_1|^2 \mathbf{1}_{\{|X_1| \geq \epsilon\sqrt{n}\}}\right]}{n\epsilon^2}$$

A factor of n cancels from numerator and denominator; by the dominated convergence theorem, the left-hand side tends to 0 as $n \to \infty$. (Or one can note that we proved $\lim_{n\to\infty} \mathbb{P}\left(|X_1| > \epsilon\sqrt{n}\right) = 0$ in problem 2.)

8. (a) For any p, $\mathbb{P}(D_p) = \mathbb{P}(X \in \{kp;\, k \geq 1\}) = (1/p^s)\sum_{1}^{\infty} k^{-s}/\xi(s) = p^{-s}$. If $p_1, \ldots, p_n$ are distinct prime numbers, then

$$\bigcap_{1}^{n} D_{p_i} = \{kp_1 \cdots p_n;\, k \geq 1\}.$$

Hence

$$\mathbb{P}\Big(\bigcap_1^n D_{p_i}\Big) = \sum_{k=1}^{\infty} \frac{(p_1 \cdots p_n)^{-s} k^{-s}}{\xi(s)} = \prod_1^n p_i^{-s}.$$

It follows that $D_{p_1}, \ldots, D_{p_n}$ are independent. (Note: If $A_1, \ldots, A_n$ are independent events, it can be proved by induction that $\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_n}$ are independent random variables, or equivalently, that the algebras $\{A_1, A_1^c, \emptyset, \Omega\}, \ldots, \{A_n, A_n^c, \emptyset, \Omega\}$, are independent.)

(b) Let $\mathcal{S}$ be the set of prime numbers. From independence (the independence of $D_1, D_2, \ldots$ entails that of the complements $D_1^c, D_2^c, \ldots$)

$$\prod_{p \in \mathcal{S}} (1 - p^{-s}) = \prod_{p \in \mathcal{S}} \mathbb{P}(D_p^c) = \mathbb{P}\Big(\bigcap_{p \in \mathcal{S}} D_p^c\Big) = \mathbb{P}(X = 1) = \frac{1}{\xi(s)}.$$

(c) There was a typo in the statement of the problem. The desired answer should be $\xi^{-1}(2s)$. The same proof as in (a) shows, that the events in the family $\{D_{p^2};\, p \text{ is prime}\}$ are independent. Since n^2 does not divide X for every $n > 1$ ($\not|$ if and only if p^2 does not divide X for every prime number p,

$$\mathbb{P}\Big(n^2 \not| \, X \text{ for every } n > 1\Big) = \mathbb{P}\Big(\bigcap_{p \in \mathcal{S}} D_{p^2}^c\Big) = \prod_{p \in \mathcal{S}} (1 - p^{-2s}) = \frac{1}{\xi(2s)}.$$

(d) Let $E_p = \{p|X, p|Y\}$. Then, using the independence of X and Y, and the result of (a), $\mathbb{P}(E_p) = p^{-2s}$ and $\{E_p;\, p \in \mathcal{S}\}$ are independent. Thus

$$\mathbb{P}\Big(H = 1\Big) = \mathbb{P}\Big(\bigcap_{p \in \mathcal{S}} E_p^c\Big) = \prod_{p \in \mathcal{S}} (1 - p^{-2s}) = \xi^{-1}(2s).$$

Next observe that

$$\mathbb{P}\Big(X/n = k \mid n \text{ divides } X\Big) = \frac{\mathbb{P}(X = nk)}{\mathbb{P}(n \text{ divides } X)} = \frac{(nk)^{-s}/\xi(s)}{n^{-s}} = \frac{k^{-s}}{\xi(s)} = \mathbb{P}(X = k).$$

Thus the conditional distribution of X/n given that n divides X is the same as that of X. A similar statement is true for Y. Because X and Y are independent,

$$\mathbb{P}(H = n) = \mathbb{P}\Big(\text{g.c.d.}(X/n, Y/n) = 1 \mid n \text{ divides } X \text{ and } Y\Big) n^{-2s} = \mathbb{P}(H = 1) n^{-2s} = \frac{n^{-2s}}{\xi(2s)}.$$

9. Since $E[X] = E[X\mathbf{1}_{\{X > bE[X]\}}] + E[X\mathbf{1}_{\{X \le bE[X]\}}] \le E[X\mathbf{1}_{\{X > bE[X]\}}] + bE[X]$, it follows that $(1-b)E[X] \le E[X\mathbf{1}_{\{X > bE[X]\}}]$. By Cauchy-Schwarz,

$$((1-b)E[X])^2 \le \mathbb{P}\left(X > bE[X]\right) E[X^2].$$

10. For any x and y, the Mean Value Theorem implies that $|e^x - e^y| \leq |x-y|(e^x + e^y)$. Also for any n and any $\epsilon > 0$, there is a constant K_ϵ such that $|x|^n \leq K_\epsilon(e^{\epsilon x} + e^{-\epsilon x})$. Thus, for example, if $|h| < \epsilon$, then since $e^{(t+h)x} \leq e^{(t+\epsilon)x} + e^{(t-\epsilon)x}$ for all x,

$$| \frac{e^{(t+h)x} - e^{tx}}{h} | \leq |x|(e^{(t+h)x} + e^{tx}) \leq K_\epsilon(e^{\epsilon x} + e^{-\epsilon x}) \left(e^{(t+\epsilon)x} + e^{(t-\epsilon)x} + e^{tx}\right).$$

It follows easily that

$$| \frac{e^{(t+h)x} - e^{tx}}{h} | \leq 5K_\epsilon \left(e^{(t+2\epsilon)x} + e^{(t-2\epsilon)x}\right).$$

Let $\mathcal{D} = \{t; M_X(t) < \infty\}$. If t is in the interior of of $\mathcal{D}$, then for small enough ϵ, $t + 2\epsilon$ and $t - 2\epsilon$ are also in $\mathcal{D}$, and hence $e^{(t+2\epsilon)X}$ and $e^{(t-2\epsilon)X}$ both have finite expectation. Therefore, by dominated convergence, using the previous inequality,

$$\lim_{h\to 0} \frac{M_X(t+h) - M_X(t)}{h} = \lim_{h\to 0} E\Big[\frac{e^{(t+h)X} - e^{tX}}{h}\Big] = E\Big[\frac{d}{dt} e^{tX}\Big] = E[Xe^{tX}],$$

(and $E[|X|e^{tX}] < \infty$) for t in the interior of $\mathcal{D}$. This proves that M_X is differentiable in the interior of $\mathcal{D}$. Using similar arguments one can prove by induction that M_X admits derivatives of all order in the interior of $\mathcal{D}$.

11. The event that $\lim_{n\to\infty} X_n$ exists is a tail event, so it has probability 0 or probability 1, by Kolmogorov's zero-one law.

Consider the event $\{\liminf_{n\to\infty} X_n \geq b\}$. This event is also a tail event, so for any b it has probability 0 or probability 1. Let

$$a = \sup\{b; \mathbb{P}(\liminf_{n\to\infty} X_n \geq b) = 1\}.$$

(If there is no b such that $\mathbb{P}(\liminf_{n\to\infty} X_n \geq b) = 1$, $a = -\infty$.) We claim that

$$\mathbb{P}\Big(\liminf_{n\to\infty} X_n = a\Big) = 1.$$

Indeed, when s is finite,

$$\mathbb{P}\Big(\liminf_{n\to\infty} X_n = a\Big) = \lim_{k\to\infty} \mathbb{P}\Big(a - 1/k < \liminf_{n\to\infty} X_n < a + 1/k)\Big) = 1,$$

by definition of a. The cases in which $a = -\infty$ or $a = \infty$ are handled easily. Since

$$\{\liminf_{n\to\infty} X_n = a\} \cap \{\lim_{n\to\infty} X_n \text{ exists}\} \subset \{\lim_{n\to\infty} X_n = a\},$$

$$\text{we see that} \quad \mathbb{P}\Big(\lim_{n\to\infty} X_n = a\Big) = 1, \quad \text{if} \quad \mathbb{P}\Big(\lim_{n\to\infty} X_n \text{ exists}\Big) = 1.$$