Levy's Construction of Brownian Motion

This discussion is based heavily on the treatment in the book *Excursions of Markov Processes*, by Blumenthal. In what follows, the notation $N(\mu, \sigma^2)$ indicates a normal random variable with mean μ and variance σ^2.

Standard Brownian motion is, by definition, a stochastic process $\{W_t; t \geq 0\}$, that satisfies:

(i) $W_0 = 0$ almost surely;

(ii) $t \to W_t(\omega)$ is continuous almost surely;

(iii) for every $0 \leq s, t$, $W_{t+s} - W_t$ is a normal random variable with mean zero and variance $t - s$ and is independent of $\sigma(W_u;\, u \leq t)$.

The difference $W_{t+s} - W_t$ is called an increment of W. A process X, which like Brownian motion, has the properties that for all positive s and t, $X_{t+s} - X_t$ is independent of $\sigma(X_u, u \leq s)$, and that the distribution of $X_{t+s} - X_t$ is the same for all $t \geq 0$ is said to have independent, stationary increments. Brownian motion has independent, stationary, *normal* increments. Since, for any $0 \leq t_1 < t_2 < \cdots < t_k$, $(W_{t_1}, \ldots, W_{t_n})$ is a linear transformation of the vector $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$ of independent normal random variables, $W_{t_1}, \ldots, W_{t_n}$ are jointly normal. This means that Brownian motion is a Gaussian process. Actually, the hypothesis in (iii) that the increments be normal is redundant. It can be shown that a zero mean process Y_t with $Y_0 = 0$ and with independent stationary increments must be Gaussian if it has continuous paths. This result ultimately relies on the Central Limit Theorem, but will not be proved here.

The distribution of a normal (Gaussian) random vector is completely determined by its mean vector and covariance matrix. This implies that the finite dimensional distributions of a Gaussian random process—the distributions of the random vectors $(X_{t_1}, \ldots, X_{t_n})$ at finite collections of times—is determined by the functions $t \to E[X_t]$ and $(t, s) \to \mathrm{Cov}(X_t, X_s)$. If W is a Brownian motion,

$$E[W_t] = 0 \quad \text{and} \quad \mathrm{Cov}(W_t, W_s) = E[W_t W_s] = t \wedge s \ (:= \min\{s, t\}) \quad \text{for all } s, t \geq 0 \tag{1}$$

Therefore, if a Gaussian process has these mean and covariance functions, it is a Brownian motion. This provides an easy-to-use tool for checking that a process is Brownian.

Proving the formula for the covariance function in (1) is easy. Let $t > s$. By the definition of Brownian motion $W_t - W_s$ is independent of W_s. Thus

$$E[W_t W_s] = E[(W_t - W_s + W_s)W_s] = E[(W_t - W_s)W_s] + E[W_s^2] = s,$$

since the covariance of independent random variable is zero and W_s is $N(0, s)$.

There is one detail to care of in this picture, however. The independent increment property, if it holds, certainly implies $\mathrm{Cov}(W_t, W_s) = t \wedge s$. We have not yet proved though the continuous time processes with independent increments exist.

Exercise. Prove that a Gaussian process satisfying (1) has independent increments. Hint: proving that $W_{t+s} - W_t$ is independent of $W_{u_1}, \ldots, W_{u_k}$ where $u_1, \ldots, u_k \leq t$ is straightforward. Use this and general principles to pass to the independence of $W_{t+s} - W_s$ and $\sigma(W_u;\, u \leq t)$.

We must make sure that the definition of Brownian motion is not an empty one. It is necessary to show that there is indeed a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process $\{W_t\}_{t \geq 0}$ defined on it satisfying all the conditions defining a Brownian motion. Using general theory, it is not hard to construct a probability space and a process W satisfying just (i) and (iii). What is more subtle is to show that one can construct such a process with continuous paths. The following, very elegant construction is due to Lévy. It will use a very simple observation.

Lemma 1 *Let Z_1 and Z_2 be independent normal random variables with mean zero variance σ^2. Then*

$$Z_1 + Z_2 \ \text{ and } \ Z_1 - Z_2 \ \text{ are independent, } N(0, 2\sigma^2) \text{ random variables.}$$

The proof is simple. We know sums of independent normals are normal and that the variance of the sum is the sum of variances. To check independence, merely note that $E[(Z_1 + Z_2)(Z_1 - Z_2)] = E[Z_1^2] - E[Z_2^2] = 0$.

Levy's construction begins with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there are defined a countable number of independent $N(0, 1)$ random variables. Such a space exists by the Kolmogorov extension theorem–see Theorem G.1 of Chapter 2– or the construction of an infinite product of probability measures–see Theorem 11 of Chapter 1 of the class notes. Given this space, let $\{Y_q; q \in \mathcal{D}\}$ denote these random variables, where, for the index set $\mathcal{D}$ we have chosen the set of dyadic rationals of $[0, 1]$; that is

$$\mathcal{D} := \bigcup_{n=0}^{\infty} \mathcal{D}_n \quad \text{where} \quad \mathcal{D}_n = \{\frac{k}{2^n}; 0 \leq k \leq 2^n\}.$$

Notice that the sequence $\{\mathcal{D}_n\}$ is increasing.

Using the random variables $\{Y_q; q \in \mathcal{D}\}$, we will construct a sequence $\{W_t^n; 0 \leq t \leq 1\}$ such that

(a) For every n, $W_\cdot^n$ is a Gaussian process on $[0,1]$ with continuous paths.

(b) The distribution of $\{W_t^n;\ t \in \mathcal{D}_n\}$—that is, the distribution of the random vector $(W_0^n, W_{1/2^n}^n, \dots, W_{k/2^n}^n, \dots, W_1^n)$— matches that required of a Brownian motion. Equivalently, $W_0^n = 0$ and for each k,

$$W_{k+1/2^n}^n - W_{k/2^n}^n \text{ is } N(0, \tfrac{1}{2^n}) \text{ and independent of } \sigma(W_u^n; u \in \mathcal{D}_n, u \leq k/2^n). \tag{2}$$

(This is equivalent, because $\{W_t^n;\ t \in \mathcal{D}_n\}$ can be reconstructed from these increments.)

(c) $W_t^{n+1} = W_t^n$ for $t \in \mathcal{D}_n$.

(d) Let $C_0[0,1] = \{f; f : [0,1] \to \mathbb{R} \text{ is continuous}\}$ and endow it with the metric $\|f\| = \sup_{t\in[0,1]} |f(t)|$. Then $\{W^n\cdot\}$, interpreted as a sequence of random maps into $C_0[0,1]$ is Cauchy; that is, $\lim_{m,n\to\infty} \|W_\cdot^n - W_\cdot^m\| = 0$.

We first show how this leads to the construction of a Brownian motion. The space $C_0[0,1]$ is complete. Thus, because of D, there is a set $A \subset \Omega$, with $\mathbb{P}(A) = 1$, such that for $\omega \in A$, there exists a continuous path $\{W_t(\omega); 0 \leq t \leq 1\}$ with

$$\lim_{n\to\infty} \|W_\cdot^n(\omega) - W_\cdot(\omega)\| = 0$$

On A^c, define $W_t(\omega) \equiv 0$. We claim that this W is a Brownian motion on the time interval $[0,1]$, and we prove this now from (a)-(d). First, it is clear by construction that the paths of W_t are continuous. Thus, according to the remark after (1), we only need to check that W is a Gaussian process with zero mean and covariance function $E[W_tW_s] = t \wedge s$. To this end let $0 \leq s_1 < s_2 < \cdots < s_m \leq 1$. Suppose first that these are all dyadic rationals. Then there is some n so that they are all in $\mathcal{D}_n$ and $W_{s_i} = W_{s_i}^n$ for all i. Hence, by (b), $W_{s_1}, \dots, W_{s_n}$ are jointly normal with the means and covariances of a Brownian motion. In particular, if $\lambda \in \mathbb{R}^n$, the joint characteristic function is

$$E\Big[\exp\{i\sum_1^m \lambda_i W_{s_i}\}\Big] = \exp\{-\frac{1}{2}\sum_{i,j=1}^m \lambda_i\lambda_j s_i \wedge s_j\}. \tag{3}$$

Now suppose that $0 \leq s_1 < s_2 < \cdots < s_m \leq 1$ is general. Then choose a sequence $\{(s_1^k, \dots, s_n^k)\}_{k\geq 0}$, each vector of which consists of dyadic rationals, such that $s_i^k \to s_i$ as $k \to \infty$ for all i. Now take limits in (3), replacing W_{s_i} in the left-hand side by $W_{s_i^k}$. By dominated convergence and continuity of paths it follows that (3) is true for any $0 \le s_1 < s_2 < \cdots < s_m \le 1$, which proves that $W_{s_1}, \ldots, W_{s_n}$ are jointly normal with mean zero and covariance $E[W_t W_s] = t \wedge s$. This completes the proof that W is a Brownian motion.

Now we come to the details of Levy's construction. First we set

$$W_t^0 := tY_1, \quad 0 \le t \le 1.$$

Since $W_0 = 0$ and W_1^0 is $N(0,1)$, this indeed coincides with the distribution that a Brownian motion would have restricted to $\{0,1\}$. Next, define W^1 as follows:

(1) $W_t^1 = W_t^1$ for $t \in \mathcal{D}_0$, namely, $W_0^1 = W_0^0 = 0$ and $W_1^1 = W_1^1 = Y_1$.
(2) Let $W_t^1 = W_t^0 + \frac{1}{2}Y_t = \frac{1}{2}Y_1 + \frac{1}{2}Y_t$ for $t \in \mathcal{D}_1 - \mathcal{D}_0 = \{\frac{1}{2}\}$.
(3) Define W^1 by linear interpolation between these points.

Observe that $W_1^1 - W_{1/2}^1 = (1/2)Y_1 - (1/2)Y_{1/2}$. Since $W_{1/2}^1 = \frac{1}{2}Y_1 + \frac{1}{2}Y_t$, We are exactly in the situation of Lemma 1 with $(Z_1, Z_2) = (1/2)(Y_1, Y_{1/2})$. It follows that, $W_{1/2}^1$ and $W_1^1 - W_{1/2}^1$ are independent $N(0,1/2)$ random variables, exactly as claimed in (b) for $n = 1$.

The construction proceeds inductively in a way reminiscent of fractal definitions. Assume that W^{n-1} has been constructed and satisfies (b). Given W^{n-1} we let W^n be equal to W^{n-1} on $\mathcal{D}_{n-1}$ and we repeat a scaled version of the procedure in (2) above in every subinterval defined by the points of $\mathcal{D}_{n-1}$: mathematically:

(1) Set $W_t^n = W_t^{n-1}$ for $t \in \mathcal{D}_{n-1}$.
(2) Set $W_t^n = W_t^{n-1} + \frac{1}{2^{(n+1)/2}} Y_t$, for $t \in \mathcal{D}_n - \mathcal{D}_{n-1}$.
(3) Define W_t^n for $t \in [0,1] - \mathcal{D}_n$ by linear interpolation.

It is clear from induction that for each n, W^n, as a process, is determined by $\{Y_q;\ q \in \mathcal{D}_n\}$. It is also clear that each W^n is Gaussian, because it is built from W^{n-1} and independent Gaussian processes by linear operations. Each W^n is continuous by definition. Hence property (a) is true for W^n. Property (c) is also a direct consequence of the construction.

To prove (b), let us take a closer look at item 2. A point $t \in \mathcal{D}_n - \mathcal{D}_n$ is midway between two successive points, call them $r < s$, of $\mathcal{D}_{n-1}$. We need to show that $W_t^n - W_2^n$ is $N(0, 1/2^n)$ and is independent of $\sigma(W_u^n; u \le r, u \in \mathcal{D}_n)$. We need also to show that $W_s^n - W_t^n$ is $N(0, 1/2^n)$ and is independent of $W_t^n - W_2^n$ and of $\sigma(W_u^n; u \le r, u \in \mathcal{D}_n)$; because $W_t^n = W_r^n + (W_t^n - W_r^n)$ it will then follow that $W_s^n - W_t^n$ is independent of $\sigma(W_u^n; u \le t, u \in \mathcal{D}_n)$, because

$\{W^n_u; u \le t, u \in \mathcal{D}_n\} = \{W^n_u; u \le r, u \in \mathcal{D}_n\} \cup \{W^n_t\}$. This will complete the proof of (b) for W^n and finish the induction step.

A simple calculation shows that

$$\begin{aligned} W^n_t - W^n_r &= \frac{1}{2}(W^{n-1}_s - W^{n-1}_r) + \frac{1}{2^{(n+1)/2}} Y_t, \quad \text{and} \\ W^n_s - W^n_t &= \frac{1}{2}(W^{n-1}_s - W^{n-1}_r) - \frac{1}{2^{(n+1)/2}} Y_t \end{aligned}$$

Everything follows from this. It fits exactly in the scheme of Lemma 1. The random variables $(1/2^{(n+1)/2} Y_t$ and $\frac{1}{2}(W^{n-1}_s - W^{n-1}_r)$ are independent and both have mean zero and variance $1/2^{n+1}$. It follows that $W^n_t - W^n_r$ and $W^n_s - W^n_t$ are independent $N(0, 1/2^n)$ random variables.

By induction, $W^{n-1}_s - W^{n-1}_r$ is independent of $\sigma(W^{n-1}_u; u \le r, u \in \mathcal{D}_{n-1})$ and Y_t is independent of Y_q for all other $q \in \mathcal{D}$. Since $\{W^n_u; u \le r, u \in \mathcal{D}_n\}$ is defined completely in terms of $\{W^{n-1}_u; u \le r, u \in \mathcal{D}_{n-1}\}$ and $\{Y_u, u \in \mathcal{D}_n, u \le r\}$, it follows that $W^n_t - W^n_r$ and $W^n_s - W^n_t$ are independent of $\sigma(W^n_u; u \le r, u \in \mathcal{D}_n)$, and we are done.

Finally, we must verify (d). It is clear from the construction that

$$\|W^n_\cdot - W^{n-1}_\cdot\| = \frac{1}{2^{(n+1)/2}} \sup_{q \in \mathcal{D}_n - \mathcal{D}_{n-1}} |Y_q|.$$

For a $N(0,1)$ random variable Y, $\mathbb{P}(Y > x) \le e^{-tx} E[e^{tX}] = e^{-tx + x^2/2}$ and optimizing over t gives $\mathbb{P}(Y > x) \le e^{-x^2/2}$, and hence, by symmetry of the standard normal, $\mathbb{P}(|Y| > x) \le 2e^{-x^2/2}$. Therefore, since $|\mathcal{D}_n - \mathcal{D}_{n-1}| = 2^{n-1}$

$$\begin{aligned} \mathbb{P}\Big(\|W^n_\cdot - W^{n-1}_\cdot\| > \frac{n^2}{2^{(n+1)/2}}\Big) &\le 2^{n-1} \mathbb{P}\Big(\frac{1}{2^{(n+1)/2}} |Y_1| > \frac{n^2}{2^{(n+1)/2}}\Big) \\ &\le 2^n e^{-n^2/2} \end{aligned}$$

Because $\sum_1^\infty 2^n e^{-n^2/2} < \infty$, the Borel-Cantelli lemma implies

$$\mathbb{P}\Big(\|W^n_\cdot - W^{n-1}_\cdot\| > \frac{n^2}{2^{(n+1)/2}} \ \text{ i.o.}\Big) = 0.$$

And since $\displaystyle\sum_1^\infty \frac{n^2}{2^{(n+1)/2}} < \infty$, it follows that

$$\sum_{n=1}^\infty \|W^n_\cdot - W^{n-1}_\cdot\| < \infty \quad \text{almost surely.}$$

This implies $\lim_{n,m\to\infty} \|W_\cdot^n - W_\cdot^{n-1}\| = 0$ almost surely and finishes the proof.

We have now constructed Brownian motion on the time interval $[0,1]$. To construct Brownian motion on $[0,\infty)$ take a probability space that supports a countable sequence $\{B_t^k; 0 \le t \le 1\}$ of independent Brownian motions defined on $[0,1]$. Let

$$W_t = (\sum_1^m B_1^k) + B_{t-m}^{m+1}, \qquad \text{if } m \le t < m+1.$$

Thus each B^m supplies the increments $W_t - W_m$ for $m \le t \le m+1$. It is easy to check that W is a Brownian motion.