

Lévy Processes, Math 642:592, Spring 2008

1. Lévy Processes and Poisson processes.

Definition: A function $f : [0, \infty) \rightarrow \mathbb{R}$ is càdlàg if it is right-continuous at all $t \in [0, \infty)$ and if the limit from the left

$$f(t-) \triangleq \lim_{s \uparrow t, s < t} f(s)$$

exists and is finite for all $t > 0$. Càdlàg functions are called R -functions in RW. The acronym “càdlàg” is from the French *continu à droite, limites à gauche*. Some authors use the English translation *corlol*.

Definition: A Lévy processes is a stochastic processes that starts at zero, has stationary, independent increments, and has càdlàg sample paths.

It is a technical detail, but the definition implies every path should be càdlàg. In discussion we loosen this to almost-surely càdlàg. The difference is minor; if the paths of X are only almost surely càdlàg and \mathcal{N} is a set of measure zero off of which $t \rightarrow X(t)(\omega)$ is càdlàg, define $X'(t)(\omega) \equiv X(t)(\omega)$ for $\omega \notin \mathcal{N}$, and $X'(t) \equiv 0$ for $\omega \in \mathcal{N}$. The processes X and X' are *indistinguishable* in the sense that $\mathbb{P}(X'(t) = X(t), \forall t > 0) = 1$.

Our first object is to define the primary example, the Poisson process. For this, let us first recall what a Poisson random variable is. Z has the Poisson(ν) distribution (where $\nu > 0$) if

$$\mathbb{P}(Z = k) = \frac{\nu^k}{k!} e^{-\nu}, \quad k = 0, 1, 2, \dots$$

In this case, $\mathbb{E}[Z] = \nu$, and $\text{Var}(Z) = \nu$, as well. We shall often identify a random variable as being Poisson by calculating its characteristic function: Z is Poisson(ν) if and only if

$$\mathbb{E}[e^{iuZ}] = e^{\nu(e^{iu} - 1)}, \quad u \in \mathbb{R}.$$

Mark this formula well. It is behind a lot of the properties of a Poisson. For example, you may verify the following:

Exercise. if Z_1 is Poisson(ν_1) and Z_2 is Poisson(ν_2) and if they are independent, $Z_1 + Z_2$ is Poisson($\nu_1 + \nu_2$).

Exercise. Let ξ_1, ξ_2, \dots be independent, identically distributed Bernoulli(p) random variables: $\mathbb{P}(\xi = 1) = p$, $\mathbb{P}(\xi_i = 0) = 1 - p$. Let Z be an independent Poisson(ν) random variable. Let

$$Y = \sum_{i=1}^Z \xi_i.$$

Then Y and $Z - Y$ are Poisson($p\nu$) and Poisson($(1 - p)\nu$), respectively, and Y and $Z - Y$ are independent. (The last, somewhat suprising statement, about independence, should be done by conditioning on Z to compute joint probabilities of Y and $Z - Y$ directly.)

We want to define Poisson processes with a minimum of assumptions. We first introduce the idea of a counting process.

Definition. A counting process is a process which starts at zero, and whose paths are (almost-surely) right-continuous, piecewise constant, increasing by unit jumps only. It is assumed that the process is finite-valued for all $t \geq 0$, almost surely.

Given a counting process N , let S_1 be the time of the first jump of N , S_2 the time of the second jump, etc. These are random variables, since they depend on the random path N . If there are only k jumps in the path $N(t)(\omega)$, $t \geq 0$, then $S_j(\omega) = \infty$ for all $j > k$. Conversely, consider a sequence of random variables T_1, T_2, \dots with values in $(0, \infty]$ (∞ allowed). Let

$$S_n = \sum_{i=1}^n T_i \quad \text{and assume that } \sum_{i=1}^{\infty} T_i = \infty, \text{ almost surely.}$$

Following the language of queueing theory, where counting processes arise as basic models, we think of $S_1 < S_2 < \dots$ as the times of successive arrivals to a queue and of T_1, T_2, \dots as interarrival times. Then

$$N(t) = \max \{n; S_n \leq t\}.$$

defines a counting process. This is how counting processes usually arise in practice.

Definition. A Poisson process is a Lévy process which is also a counting process.

The next theorem justifies the terminology.

Theorem 1 *A Lévy process N is a counting process if and only if there exists a $\lambda \geq 0$ such that*

$$\mathbb{E} \left[e^{iuN(t)} \right] = e^{\lambda t(e^{iu} - 1)}, \quad \text{for all } t \geq 0. \quad (1)$$

Thus, in the case that $\lambda = 0$, the paths of N are identically zero almost surely. Because of (1), when $\lambda > 0$, $N(t) - N(s)$ is Poisson($\lambda(t - s)$) for all $0 \leq s < t$.

The parameter λ is called the rate of the Poisson process. This terminology makes sense because the average number of arrivals in any interval of length t is $\mathbb{E} [N(s+t) - N(s)] = \lambda t$.

Proof: Assume that N is a Lévy process and $N(t) - N(s)$ is Poisson($\lambda(t - s)$) for any $0 \leq s < t$. Let

$$A \triangleq \{\omega; \exists t > 0, N(t) \text{ is not a non-neg. integer}\} \cup \{\omega; \exists 0 \leq s < t, \text{ such that } N(t) < N(s)\}$$

By right-continuity of the Lévy process, this is in the complement of

$$B \triangleq \{\omega; \forall t \in \mathbb{Q} \cap [0, \infty), N(t) \text{ is a non-neg. integer}\} \cup \{\omega; \forall 0 \leq s < t, s, t \in \mathbb{Q} \cap [0, \infty), N(t) \geq N(s)\}$$

However, $\mathbb{P}(B) = 1$ because $\mathbb{P}(N(t) \text{ is a non-neg. integer}) = 1$ for every $t \geq 0$ and $\mathbb{P}(N(s) \leq N(t)) = 1$ for every $0 \leq s < t$. (Here, as in many future arguments, path-regularity is needed to

make almost-sure statements about properties of a path for all $t \geq 0$.) This implies that $\mathbb{P}(A) = 0$ and hence that the sample paths of N almost-surely take values in the non-negative integers and are non-decreasing. It follows that the paths are piecewise-constant increasing only by jumps, almost-surely. By re-defining $N(t)(\omega)$ on the complement of B so that it is equivalent to the path that is identically 0 we obtain a version of N all of whose paths are right-continuous, non-decreasing and take values in the non-negative integers only. Since we have only changed N on a set of probability 0, its increments are still Poisson. We need to show that its paths increase by unit jumps only, almost-surely. It suffices to show this on any bounded interval. For convenience we consider $[0, 1]$. Let

$$M_n = \max \left\{ N\left(\frac{k}{2^n}\right) - N\left(\frac{k-1}{2^n}\right); 1 \leq k \leq 2^n \right\}$$

This sequence is non-increasing and

$$\sup \{N(t) - N(t-); 0 \leq t \leq 1\} \leq M_n, \quad \forall n.$$

Thus, the sequence of events $U_n = \{M_n \leq 1\}$ is increasing and

$$\cup_1^\infty U_n \subset \{\sup\{N(t) - N(t-); 0 \leq t \leq 1\} \leq 1\}.$$

Therefore to show that N has unit jumps almost surely on $[0, 1]$ it suffices to show that $\lim_{n \rightarrow \infty} \mathbb{P}(U_n) = 1$. But, using the independence and Poisson distribution of the increments,

$$\mathbb{P}(U_n) = [\mathbb{P}(N(2^{-n}) \leq 1)]^{2^n} = \left[e^{-\lambda/2^n} + \frac{\lambda}{2^n} e^{-\lambda/2^n} \right]^{2^n} \rightarrow 1, \quad n \rightarrow \infty,$$

as an easy calculation shows.

For the converse we will show that if N is a Lévy process which is also a counting process and if $\psi(t) \triangleq \mathbb{E} [e^{iuN(t)}]$, then $\psi(t)$ is differentiable and

$$\psi'(t) = \lambda(e^{iu} - 1)\psi(t). \tag{2}$$

The solution to this equation is $\psi(t) = \psi(0)e^{\lambda t(e^{iu}-1)} = e^{\lambda t(e^{iu}-1)}$, since $\psi(0) = 1$, and so showing (2) will complete the proof.

The proof requires several steps. The first step is to show that there is a $\lambda \geq 0$ such that

$$\mathbb{P}(N(t) = 0) = e^{-\lambda t} \tag{3}$$

For this, define $U(t) = \mathbb{P}(N(t) = 0)$. By independence and stationarity of increments

$$U(t+s) = \mathbb{P}(N(t+s) = 0) = \mathbb{P}(N(t+s) - N(t) = 0, N(t) = 0) = \mathbb{P}(N(s) = 0)\mathbb{P}(N(t) = 0) = U(t)U(s). \tag{4}$$

Exercise. Show that U is right continuous, that $U(1) > 0$ and that the solution to (4) must be of the form

$$U(t) = e^{-\lambda t}, \quad \text{where } \lambda = -\ln U(1).$$

Note that if $\lambda = 0$, the sample paths of N are identically 0 and the proof is done. If $\lambda > 0$, we proceed with the second step, which is to show that

$$\mathbb{P}(N(h) \geq 2) = o(h), \quad \text{as } h \downarrow 0. \quad (5)$$

Recall that this means $\lim_{h \downarrow 0} \mathbb{P}(N(h) \geq 2)/h = 0$. This fact is a consequence of the independence and stationarity of increments, combined with the fact that jumps are of at most unit size. The fact that jumps are of unit size implies

$$1 = \lim_{n \rightarrow \infty} \mathbb{P}(\max\{N(k/n) - N((k-1)/n); 1 \leq k \leq n\} \leq 1).$$

Using independence and stationarity of increments, this implies

$$1 = 1 - \lim_{n \rightarrow \infty} [1 - \mathbb{P}(N(1/n) \geq 2)]^n.$$

Exercise. Derive the conclusion: $\lim_{n \rightarrow \infty} n\mathbb{P}(N(1/n) \geq 2) = 0$. Verify that for $1/(n+1) < h < 1/n$, $h^{-1}\mathbb{P}(N(h) \geq 2) \leq (n+1)\mathbb{P}(N(1/n) \geq 2)$ and hence conclude (5).

Now observe that by independence and stationarity of increments $\psi(t+h) = \psi(t)\psi(h)$, when $t \geq 0, h > 0$. However,

$$\begin{aligned} \psi(h) &= \mathbb{E} \left[e^{iuN(h)} \right] = 1 \cdot \mathbb{P}(N(h)=0) + e^{iu} [1 - \mathbb{P}(N(h)=0) - \mathbb{P}(N(h) \geq 2)] + \mathbb{E} \left[e^{iuN(h)} \mathbf{1}_{N(h) \geq 2} \right] \\ &= e^{-\lambda h} + e^{iu} [1 - e^{-\lambda h}] + o(h). \end{aligned}$$

(From this we can conclude immediately that $\psi(t)$ is continuous in t .) It follows that

$$\frac{\psi(t+h) - \psi(t)}{h} = \frac{[e^{iu} - 1][1 - e^{-\lambda h}]}{h} \psi(t) + \frac{o(h)}{h}$$

and hence

$$\lim_{h \downarrow 0} \frac{\psi(t+h) - \psi(t)}{h} = \lambda(e^{iu} - 1).$$

A simple variation shows also that $\lim_{h \uparrow 0} \frac{\psi(t+h) - \psi(t)}{h} = \lambda(e^{iu} - 1)$. This completes the proof. \diamond

This discussion was drawn with modification from *Probability*, by Breiman, and from *Probability and Measure*, by Billingsley.

2. A property of Poisson processes.

We discuss an independence property alluded to in lecture.

Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a given filtration. A Poisson process that is adapted to \mathbb{F} and that satisfies in addition that $N(t+s) - N(s)$ is independent of \mathcal{F}_s for all $0 \leq s$ and $0 < t$, is called an \mathbb{F} -Poisson process.

Theorem 2 *If N_1 and N_2 are two \mathbb{F} -Poisson processes and if they admit no simultaneous jumps with probability 1, then they are independent.*

This is a surprising fact that is most easily proved using stochastic calculus. However, it can be attacked using the methods of the proof of Theorem 1. We need to use the fact that if

$$\mathbb{E} \left[e^{iu_1 Z_1 + iu_2 Z_2} \right] = e^{\nu_1(e^{iu_1} - 1)} e^{\nu_2(e^{iu_2} - 1)},$$

then Z_1 and Z_2 are independent Poisson(ν_1) and Poisson(ν_2) respectively.

The real meaning of having N_1 and N_2 both be \mathbb{F} -Poisson processes with respect to the same filtration is that $N_1(t+s) - N_1(t)$ and $N_2(t+s) - N_2(t)$ are each independent of $\sigma\{N_i(r); r \leq t, i = 1, 2\}$.

Following is a series of exercises to prove Theorem 2. Note that to say that two processes X_1 and X_2 are independent means that every event in $\sigma\{X_1(t); t \geq 0\}$ is independent of every event in $\sigma\{X_2(t); t \geq 0\}$. For this, monotone convergence techniques show that it suffices to prove that for any finite set of times $0 \leq t_1 < \dots < t_n$ ($X_1(t_1), \dots, X_1(t_n)$) and ($X_2(t_1), \dots, X_2(t_n)$) are independent.

Exercise. Show that, under the assumptions of the Theorem, $\mathbb{P}(N_1(h) \geq 1, N_2(h) \geq 1) = o(h)$.

Exercise. Letting λ_i denote the rate of N_i , show that for all $t \geq 0$,

$$\mathbb{E} \left[e^{iu_1 N_1(t) + iu_2 N_2(t)} \right] = e^{t\lambda_1(e^{iu_1} - 1)} e^{t\lambda_2(e^{iu_2} - 1)},$$

and conclude that $N_1(t)$ and $N_2(t)$ are independent for each $t \geq 0$.

Exercise. Show that N_1 and N_2 are independent processes.

3. Constructing a Poisson process

We have not shown that a Poisson process actually exists. Here is an elementary construction. Let T_1, T_2, \dots be a sequence of independent identically distributed exponential(λ) random variables. Thus, $\mathbb{P}(T_i \leq 0) = 0$ and $\mathbb{P}(T_i > x) = e^{-\lambda x}$, $x > 0$. These will be the interarrival times. The time of the arrival n is then $S_n = \sum_1^n T_i$. Let $N(t)$ count the number of arrivals by time t : $N(t) = \max\{n; S_n \leq t\}$.

Theorem 3 *N is a Poisson process with rate λ .*

Proof. N is obviously a counting process. That $N(t)$ is Poisson can be computed directly from the observation that S_n has a gamma distribution with parameters n and λ : the density of S_n is $f_n(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}$.

Exercise. Use this to prove that N is a Poisson process with rate λ . First prove that $N(t)$ is Poisson(λt), for any $t > 0$. Then for any $t \geq 0$, $s > 0$, compute $\mathbb{P}(N(t+s) - N(t) = k, N(t) = l)$ —use conditioning. (This is one approach; you may think of others.)

There is another construction of a Poisson process that is very useful. Fix $\tau > 0$. We will construct a Poisson process with rate λ up to time τ . By using this construction to create a

sequence of independent Poisson processes with rate λ on the successive intervals $(\tau, 2\tau]$, $(2\tau, 3\tau]$, etc., and piecing them together, one will obtain a Poisson process defined for $t \geq 0$.

Let Z be a $\text{Poisson}(\lambda\tau)$ random variable. Let ρ_1, ρ_2, \dots be a sequence of independent random variables uniformly distributed on $(0, \tau)$, all of them independent of Z . Define

$$M(t) = \sum_1^Z \mathbf{1}_{[0,t]}(\rho_i), \quad 0 \leq t \leq \tau.$$

(When $Z = 0$, $M(t) = 0$ for all $0 \leq t \leq \tau$.) N is clearly a counting process. At each t , $M(t)$ counts the number of points among ρ_1, \dots, ρ_Z which fall in the interval $[0, t]$. Thus the arrival times $S_1 < S_2 < \dots < S_Z$ are the values of $\{\rho_1, \dots, \rho_Z\}$ arranged in increasing order. In the official terminology of probability and statistics, $\{S_1, S_2, \dots, S_Z\}$ are the *order statistics* of $\{\rho_1, \dots, \rho_Z\}$. In words, this construction amounts to, first, choosing how many arrivals Z will occur in $[0, \tau]$, according to a Poisson distribution, and then independently choosing Z points in $[0, \tau]$ by independent random sampling of a uniform distribution on $[0, \tau]$.

Theorem 4 $\{M(t); 0 \leq t \leq \tau\}$ is a Poisson process on $[0, \tau]$ with rate λ .

This theorem has an immediate and interesting consequence.

Corollary 1 Let N be a Poisson process with rate λ . Then for every $t \geq 0$, conditional on $\{N(t) = n\}$, the arrival times (S_1, \dots, S_n) have the same distribution as the order statistics of a set n independent random variables drawn from the uniform distribution on $(0, t)$.

Proof of Theorem 4: Let $0 < t_1 < \dots < t_k = \tau$ be a partition of $[0, \tau]$. Let j_1, \dots, j_k be non-negative integers.

Exercise. Show that

$$\begin{aligned} & \mathbb{P} \left(M(t_1) = j_1, M(t_2) - M(t_1) = j_2, \dots, M(t_k) - M(t_{k-1}) = j_k \mid Z = n \right) \\ &= \frac{n!}{j_1! \cdots j_k!} \left(\frac{t_1}{\tau} \right)^{j_1} \left(\frac{t_2 - t_1}{\tau} \right)^{j_2} \cdots \left(\frac{t_k - t_{k-1}}{\tau} \right)^{j_k}, \end{aligned}$$

if $n = j_1 + \dots + j_k$ and that it equals 0 otherwise. Conclude that

$$\begin{aligned} & \mathbb{P}(M(t_1) = j_1, M(t_2) - M(t_1) = j_2, \dots, M(t_k) - M(t_{k-1}) = j_k) \\ &= \frac{(\lambda t_1)^{j_1}}{j_1!} e^{-\lambda t_1} \cdot \frac{(\lambda(t_2 - t_1))^{j_2}}{j_2!} e^{-\lambda(t_2 - t_1)} \cdots \frac{(\lambda(t_k - t_{k-1}))^{j_k}}{j_k!} e^{-\lambda(t_k - t_{k-1})} \end{aligned}$$

This completes the proof since it establishes the independence, stationarity and Poisson distribution of the increments of M . \diamond

Exercise. Let Y_1, \dots, Y_n be independent, identically distributed random variable having the uniform distribution on $(0, \tau)$. Let $0 < Z_1 < Z_2 < \dots < Z_n$ denote the values of the Y_i 's arranged in

increasing order; Z_k is called the k^{th} order statistic of (Y_1, \dots, Y_n) . One can show that the joint probability density of (Z_1, \dots, Z_n) is

$$f(z_1, \dots, z_n) = \begin{cases} \frac{n!}{\tau^n}, & \text{if } 0 < z_1 < \dots < z_n < \tau; \\ 0, & \text{otherwise.} \end{cases}$$

(This is true because the joint density of (Y_1, \dots, Y_n) is $(1/\tau)^n \mathbf{1}_{(0,1)}(y_1) \cdots \mathbf{1}_{(0,1)}(y_n)$ and all orderings of (Y_1, \dots, Y_n) are equally likely.) Give a direct proof of Corollary 1 by showing that if N is a Poisson process with rate λ , then conditional on $N(\tau) = n$, the joint distribution of (S_1, \dots, S_n) is just the same as that of (Z_1, \dots, Z_n) . One approach is to observe that the joint density at $0 < t_1 < \dots < t_n < \tau$ of (S_1, \dots, S_n) , given that $N(t) = n$, is, by differentiation of the conditional distribution measure.

$$g(t_1, \dots, t_n) = \lim_{h \downarrow 0} \frac{\mathbb{P}(\cap_1^n \{S_i \in (t_i, t_i + h]\} \mid N(\tau) = n)}{h^n}.$$

Calculate the right-hand side using the properties of a Poisson process and show that the result is the same as $f(t_1, \dots, t_n)$ stated above.

Here is an interesting consequence of Corollary 1 that we will use in the next section. Let N be a Poisson process with rate λ . Given any Borel subset A of $[0, \infty)$, let $N(A)$ count the number of arrivals of N that fall in A ; thus (the notation is a little ambiguous) $N((s, t]) = N(t) - N(s)$. Let $|A|$ be the Lebesgue measure of A .

Corollary 2 *If $|A| < \infty$ then $N(A)$ is a Poisson($\lambda|A|$) random variable. If A and B are disjoint, then $N(A)$ and $N(B)$ are independent.*

Proof. Again, we leave working out the details as exercises. First consider $A \subset [0, \tau]$. The proof is like the proof of Theorem 4, except that instead of a regular partition of $[0, \tau]$ we partitioned $[0, \tau]$ into A and its complement in $[0, \tau]$. Then use Corollary 1 to show that

$$\mathbb{P}\left(N(A) = k \mid N(\tau) = n\right) = \binom{n}{k} \left(\frac{|A|}{\tau}\right)^k \left(1 - \frac{|A|}{\tau}\right)^{n-k},$$

if $n \geq k$. Use this to conclude that $N(A)$ is Poisson($\lambda|A|$). For general A , note that $N(A) = \lim_{\tau \rightarrow \infty} N(A \cap [0, \tau])$. Show that $N(A \cap [0, \tau])$ converges in distribution to a Poisson($\lambda|A|$). A similar approach can be applied to show the independence of $N(A)$ and $N(B)$. Do the details as an exercise.

4. Compound Poisson Processes, Poisson Point Process and Poisson Random Measures.

Let N be a Poisson process with rate λ . Let ξ_1, ξ_2, \dots be a sequence of independent, identically distributed random variables having a probability law ν . That is,

$$\nu(B) = \mathbb{P}(\xi \in B), \quad B \in \mathcal{B}(\mathbb{R}) (= \text{Borel subsets of } \mathbb{R}).$$

Assume that this sequence of random variables is independent of N . Define

$$X(t) = \sum_1^{N(t)} \xi_i, \quad (6)$$

interpreting the sum as 0 if $N(t) = 0$. Such a process X is called a *compound Poisson process*. It has right-continuous, piecewise constant sample paths, but, instead of jumping upward by one unit at each arrival time, it jumps by ξ_{S_n} at the n^{th} arrival time S_n . It is easy to see that increments of X are stationary. A simple argument, conditioning on N also shows that they are independent. Hence X is a Lévy process.

Let Π be the bounded measure $\lambda\nu$. In terms of ν the characteristic function of the jump random variables is

$$\mathbb{E} \left[e^{iu\xi_i} \right] = \int_{\mathbb{R}} e^{iux} \nu(dx).$$

The following computation is basic.

$$\begin{aligned} \mathbb{E} \left[e^{iuX(t)} \right] &= \mathbb{E} \left[\mathbb{E} \left[e^{iu \sum_1^{N(t)} \xi_i} \mid N(t) \right] \right] = \mathbb{E} \left[\left(\int e^{iux} \nu(dx) \right)^{N(t)} \right] \\ &= e^{t\lambda \left(\int e^{iux} \nu(dx) - 1 \right)} = e^{t \int (e^{iux} - 1) \Pi(dx)} \end{aligned}$$

The measure Π is called the characteristic measure of X .

Exercise. (Also very basic and important) For the compound Poisson process defined in (6),

$$\mathbb{E} [X(t)] = \lambda t \mathbb{E} [\xi_i] = \lambda t \int x \nu(dx) \quad \text{and} \quad \text{Var}(X(t)) = \lambda t \mathbb{E} [\xi_i^2] = \lambda t \int x^2 \nu(dx) \quad (7)$$

These calculations are carried out by conditioning first on $N(t)$ and using the fact that the mean and variance of $N(t)$ are both λt .

Conversely, given a bounded measure Π , we can construct a compound Poisson process X satisfying

$$\mathbb{E} \left[e^{iuX(t)} \right] = e^{t \int (e^{iux} - 1) \Pi(dx)}$$

by the formula (6) by taking N to be a Poisson process with rate $\Pi(\mathbb{R})$ and letting the distribution of the ξ_i be $\mathbb{P}(\xi_i \in U) = \Pi(U)/\Pi(\mathbb{R})$.

There are equivalent ways of encoding the information in the compound Poisson process. Let $\{S_k\}$ be the sequence of arrival times of N . Imagine that in $[0, \infty) \times \mathbb{R}$ space we plot the points (S_k, ξ_k) . One way to look at this collection of random points is as a random measure. Let $\delta_{(t,x)}$ denote the Dirac measure at (t, x) ; $\delta_{(t,x)}(A) = \mathbf{1}_A(t, x)$, for Borel subsets of $[0, \infty) \times \mathbb{R}$. Then,

$$\Upsilon = \sum_k \delta_{(S_k, \xi_k)}$$

defines a random measure on the Borel subsets of $[0, \infty) \times \mathbb{R}$; $\Upsilon(A)$ counts the random number of the points (S_k, ξ_k) , $k \geq 1$ falling in A . As you might guess by now, $\Upsilon(A)$ will have a Poisson distribution. We make the following definition.

Definition. Let S be a metric space endowed with the σ -algebra $\mathcal{B}(S)$ of its Borel sets. Let μ be a σ -finite measure on $[0, \infty) \times \mathbb{R}$. A Poisson random measure Φ on $(S, \mathcal{B}(S))$ with intensity μ is a random measure such that:

- (i) $\Phi(B)$ is $\text{Poisson}(\mu(B))$ for every Borel set B such that $M(B) < \infty$;
- (ii) If B_1, \dots, B_n are disjoint, then $\Phi(B_1), \dots, \Phi(B_n)$ are independent.

Example. In the last section, we defined the random measure N on $[0, \infty)$ associated to a Poisson process, by defining $N(A)$ as the number of arrivals falling in A . Corollary 2 says that it is a Poisson random measure.

In the discussion below, we shall use m to denote Lebesgue measure on $[0, \infty)$.

Theorem 5 Υ is a Poisson random measure with intensity $\mu(A) = m \times \Pi$.

Proof. The most direct proof of this theorem uses Corollary 1. Suppose $A \subset [0, \tau] \times \mathbb{R}$. Then, conditional on $N(t) = n$,

$$\Upsilon(A) = \sum_1^N(t) \mathbf{1}_A(S_k, \xi_k)$$

will have the same distribution as

$$\sum_1^n \mathbf{1}_A(Y_i, \xi_k)$$

where Y_1, \dots, Y_n are independent random variables uniformly distributed on $(0, \tau)$ that are, in addition, independent of ξ_1, \dots . This is binomial random variable with parameters n and

$$p = \mathbb{P}((Y_1, \xi_1) \in A) = \int_{[0, \infty)} \int_{\mathbb{R}} \mathbf{1}_A(t, x) \nu(dx) \frac{1}{\tau} dt = \frac{1}{\lambda\tau} \mu(A).$$

Thus (see the proof of Corollary 2 for a similar argument)

$$\begin{aligned} \mathbb{P}(\Upsilon(A) = k) &= \sum_0^\infty \mathbb{P}\left(\Upsilon(A) = k \mid N(\tau) = n\right) \mathbb{P}(N(\tau) = n) \\ &= \sum_{n=k}^\infty \binom{n}{k} \left(\frac{\mu(A)}{\lambda\tau}\right)^k \left(\frac{\lambda\tau - \mu(A)}{\lambda\tau}\right)^{n-k} \cdot \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} \\ &= \frac{\mu^k(A)}{k!} e^{-\mu(A)} \end{aligned}$$

If A and B are disjoint and contained in $[0, \tau] \times \mathbb{R}$, then

$$\mathbb{P}\left(\Upsilon(A) = k, \Upsilon(B) = j \mid N(\tau) = n\right) = \frac{n!}{k!j!(n-k-j)!} \left(\frac{\mu(A)}{\lambda\tau}\right)^k \left(\frac{\mu(B)}{\lambda\tau}\right)^j \left(\frac{\lambda\tau - \mu(A) - \mu(B)}{\lambda\tau}\right)^{n-j-k},$$

if $n \geq k + j$, and hence

$$\begin{aligned} \mathbb{P}(\Upsilon(A) = k, \Upsilon(B) = j) &= \sum_{j+k}^{\infty} \mathbb{P}\left(\Upsilon(A) = k, \Upsilon(B) = j \mid N(\tau) = n\right) \mathbb{P}(N(\tau) = n) \\ &= \frac{\mu^k(A)}{k!} e^{-\mu(A)} \cdot \frac{\mu^j(B)}{j!} e^{-\mu(B)}. \end{aligned}$$

This proves the independence.

For general A and B , one takes limits of $\Upsilon(A \cap [0, \tau] \times \mathbb{R})$ and $\Upsilon(B \cap [0, \tau] \times \mathbb{R})$, as $\tau \rightarrow \infty$. \diamond

An equivalent way to capture this information in the compound Poisson process is to define the associated *Poisson point process*. This is a process which records at each arrival time S_k the associated jump of the compound Poisson process ξ_k . To express it we need a kind of null state, which we call here ∂ , to account for times at which there is not an arrival. For the compound Poisson process X defined above, we define the associated Poisson point process:

$$\Delta^X(t) = \begin{cases} \xi_i, & \text{if } t \in \{S_k; k \geq 1\} \text{ and } t = S_i; \\ \partial, & \text{otherwise.} \end{cases}$$

This is the Poisson measure associated to characteristic measure Π . It relates very simply to X by

$$X(t) = \sum_{s \leq t} \Delta_s^X \mathbf{1}_{\{\Delta^X(t) \neq \partial\}} \quad (8)$$

So far, we started with a compound Poisson process and constructed its associated Poisson measure and Poisson point process. Now we would like to generalize, and work backward, from the specification of an intensity measure of the form

$$\mu = m \times \Pi, \quad \text{where } m \text{ is Lebesgue measure,}$$

to an associated Poisson measure or Poisson point process. For the compound process Poisson process, the measure Π was finite. But now we will drop this assumption. Instead we shall assume that

Π is σ -finite.

This means that there is a disjoint partition $\{I_k\}$ of \mathbb{R} such $\Pi(I_k) < \infty$ for every k . To construct the associated Poisson measure and point process, we simply construct the associated processes on each k and add. Thus for each k , let N_k be a Poisson process with rate $\lambda_k = \Pi(I_k)$, and let N_1, \dots be independent. For each k , let $\xi_1^{(k)}, \xi_2^{(k)}, \dots$ be independent with distribution $\mathbb{P}(\xi_i^{(k)} \in B) = \Pi(B)/\lambda_k$. Note that $\mathbb{P}(\xi_i^{(k)} \in I_k) = 1$ so for different k the $\xi_i^{(k)}$ fall in disjoint sets. Again, let these sequences be independent for different k and assume they are all independent of the Poisson processes N_k , $k \geq 1$. Let $\{S_j^{(k)}\}$ be the sequence of arrival times for process N_k . Then

$$\Upsilon = \sum_{k \geq 1, i \geq 1} \delta_{(S_i^{(k)}, \xi_i)}$$

forms a random measure. It is a sum of the independent Poisson measures $\Upsilon_k = \sum_i \delta_{(S_i^{(k)}, \xi_i)}$ and it is not hard to show that it is a Poisson random measure with characteristic measure μ . The details are left as an exercise.

The point process associated to Υ is simply

$$\Delta^\Upsilon(t) = \begin{cases} \xi_i^{(k)}, & \text{if } t \in \{S_i^{(k)}; k \geq 1, i \geq 1\} \text{ and } t = S_i^{(k)}; \\ \partial, & \text{otherwise.} \end{cases}$$

When Π is *not* a bounded measure, the expression

$$\sum_{s \leq t} \Delta^\Upsilon(t) \mathbf{1}_{\{\Delta^\Upsilon(t) \neq \partial\}}$$

will not in general define a convergent series for any $t > 0$, and hence will not define a process, contrary to the case in which Π is bounded. However, the following will be true, by virtue of the fact that Υ is a Poisson random measure.

Lemma 1 *If A is a Borel subset of \mathbb{R} such that $\Pi(A) < \infty$, then*

$$X_A(t) \triangleq \sum_{s \leq t} \mathbf{1}_{\{\Delta^\Upsilon(t) \in A\}} \Delta^\Upsilon(t) = \int \mathbf{1}_{[0,t] \times A}(s, x) x \Upsilon(ds dx)$$

is a compound Poisson process with characteristic function

$$\mathbb{E} \left[e^{iuX_A(t)} \right] = \exp \left\{ t \int (e^{iu} - 1) \mathbf{1}_A(x) \Pi(dx) \right\}.$$

Thus, the process X_A has the same distribution as the compound Poisson process $\sum_1^{N(t)} \zeta_i$ obtained using a Poisson process $N(t)$ with rate $\Pi(A)$ and independent ζ_1, \dots that have the distribution $\mathbb{P}(\zeta_i \in B) = \Pi(B \cap A) / \Pi(A)$.

5. The structure of a general Lévy process.

Let $X(t)$ be a general Lévy process. The main theorem of this section is

Theorem 6 *For any numbers μ and σ^2 with $\sigma^2 \geq 0$ and any σ -finite measure Π on \mathbb{R} such that $\Pi(\{0\}) = 0$ and $\int (x^2 \wedge 1) \Pi(dx) < \infty$, there is a Lévy process $X(t)$ such that*

$$\mathbb{E} \left[e^{iuX(t)} \right] = \exp \left\{ t \left[i\mu u - \frac{\sigma^2 u^2}{2} + \int [e^{iux} - 1 - iux \mathbf{1}_{(-1,1)}(x)] \Pi(dx) \right] \right\} \quad (9)$$

Moreover, one can decompose $X(t)$ as the sum $X(t) = (\mu t + \sigma W(t)) + X_2(t) + X_3(t)$ of three independent Lévy process processes, where

(i) *W is a standard Brownian motion;*

(ii) X_2 is a compound Poisson process whose characteristic measure is $\Pi_2(A) = \Pi(A \cap (-1, 1)^c)$; and,

(iii) X_3 is a martingale and $\mathbb{E} \left[e^{iuX_3(t)} \right] = \exp \left\{ t \int_{(-1,1)} \left[e^{iux} - 1 - iux \right] \Pi(dx) \right\}$.

Conversely, given any Lévy process, there are μ, σ^2 ($\sigma^2 \geq 0$, and a measure Π on \mathbb{R} satisfying $\int (x^2 \wedge 1), \Pi(dx) < \infty$ and $\Pi(\{0\}) = 0$, such that (9) holds.

Remarks: 1. Notice that, using the form of the characteristic function of Brownian motion,

$$\mathbb{E} \left[e^{iu(\mu t + \sigma W(t))} \right] = \exp \left\{ t \left[i\mu u - \frac{\sigma^2 u^2}{2} \right] \right\}.$$

Likewise, by definition,

$$\mathbb{E} \left[e^{iuX_2(t)} \right] = \exp \left\{ t \left[\int_{(-1,1)^c} (e^{iux} - 1) \Pi(dx) \right] \right\}.$$

This makes sense and defines a compound Poisson process because the assumption

$$\int (X^2 \wedge 1) \Pi(dx) < \infty$$

entails that $\Pi(\int_{(-1,1)^c}) < \infty$. Because the processes W, X_2 , and X_3 are independent, the the characteristic function of their sum is just the product of their characteristic functions and hence equals the expression in (9).

2. Recall from Lecture 3 notes that a random variable Y is infinitely divisible (or that its probability distribution is infinitely divisible) if, for every positive integer n , there are independent and identically distribute random variables $Z_1^{(n)}, \dots, Z_n^{(n)}$, such that Y has the same probability distribution as $Z_1^{(n)} + \dots + Z_n^{(n)}$. The famous Lévy-Khinchine theorem that Y is infinitely divisible if and only if there exist unique $\sigma^2 \geq 0, m$, and a measure Π on \mathbb{R} satisfying $\int (x^2 \wedge 1), \Pi(dx) < \infty$ and $\Pi(\{0\}) = 0$, such that

$$\mathbb{E} \left[e^{iuY} \right] = \mathbb{E} \left[e^{iuX(t)} \right] = \exp \left\{ i\mu u - \frac{\sigma^2 u^2}{2} + \int \left[e^{iux} - 1 - iux \mathbf{1}_{(-1,1)}(x) \right] \Pi(dx) \right\}.$$

We will defer the proof of this for the moment.

Now let X be a Lévy process. Then since $X(t) = \sum_1^n X(tk/n) - X(t(k-1)/n)$, for every $n \geq 1$, and for any $t > 0$, the fact that X has independent stationary increments implies that $X(t)$ is infinitely divisible for every t . Fixing our attention on $t = 1$, the Lévy-Khinchine representations provides $m, \sigma^2 \geq 0$ and Π such that

$$\mathbb{E} \left[e^{iuX(1)} \right] = \mathbb{E} \left[e^{iuX(t)} \right] = \exp \left\{ i\mu u - \frac{\sigma^2 u^2}{2} + \int \left[e^{iux} - 1 - iux \mathbf{1}_{(-1,1)}(x) \right] \Pi(dx) \right\}.$$

The main significance of Theorem 6 is that we can go in reverse. For any infinitely divisible distribution, there is a Lévy process X such that $X(1)$ has that infinitely divisible distribution. The linearity of the argument of the exponent in (9) in t reflects and is a consequence of the stationarity and independence of increments and the right continuity of paths, because for any $s, t \geq 0$, $\mathbb{E} \left[e^{iX(t+s)} \right] = \mathbb{E} \left[e^{iX(t)} \right] \mathbb{E} \left[e^{iX(s)} \right]$.

Proof: We know how to construct a Brownian motion and we know how to construct the compound Poisson process X_2 . From Remark 1, to complete the proof it is necessary to construct $X_3(t)$, and it suffices to do this on the interval $0 \leq t \leq 1$. For the construction we will use the following fact.

Lemma 2 *If $Y(t)$ is a martingale with right-continuous paths then*

$$\mathbb{E} \left[\sup_{t \in [0, T]} Y^2(t) \right] \leq 4\mathbb{E} \left[Y^2(T) \right]. \quad (10)$$

Proof: This is a direct consequence of the martingale inequality stated in Theorem 4, equation (4) of the Lecture 2 notes for $p = 2$: for a discrete time martingale M

$$\mathbb{E} \left[\sup_{k \leq n} M_n^2 \right] \leq 4\mathbb{E} \left[M_n^2 \right].$$

For each n $\{Y(kT/n); k \geq 0\}$ defines a discrete time martingale so

$$\mathbb{E} \left[\sup_{k \leq n} Y(kT/n)^2 \right] \leq 4\mathbb{E} \left[Y(T)^2 \right].$$

Now let $n \rightarrow \infty$ and use the right-continuity of paths and monotone convergence. ◇

Proof (sketch) of Theorem 6. Let Δ_t be the Poisson point process with intensity measure $m \times \Pi$ on $[0, \infty) \times \mathbb{R}$. Let Υ be the associated Poisson random measure.

Let $\epsilon_n \downarrow 0$. For each n $\Pi(-1, 1) - (-\epsilon_n, \epsilon_n) < \infty$ by virtue of the fact that $\int (x^2 \wedge 1) \Pi(dx) < \infty$. Therefore,

$$Z_3^{(n)}(t) \triangleq \sum_{s \leq t} \mathbf{1}_{\{1 > |\Delta_t| \geq \epsilon_n\}} \Delta_t$$

defines a compound Poisson process. Using the formulas for mean and variance of a compound Poisson process stated above in (7),

$$\mathbb{E} \left[Z^{(n)}(t) \right] = t \int \mathbf{1}_{\{1 > |x| \geq \epsilon_n\}} x \Pi(dx) \quad \text{and} \quad \text{Var}(Z^{(n)}(t)) = t \int \mathbf{1}_{\{1 > |x| \geq \epsilon_n\}} x^2 \Pi(dx). \quad (11)$$

It follows that for each n

$$X_3^{(n)}(t) \triangleq Z_3^{(n)}(t) - t \int \mathbf{1}_{\{1 > |x| \geq \epsilon_n\}} x \Pi(dx).$$

is a martingale with respect to the filtration \mathbb{F} where \mathcal{F}_t is the σ -algebra generated by $\{\Upsilon(A); A \in [0, t] \times \mathbb{R}\}$. Moreover it has càdlàg paths for each n . From the martingale inequality of Lemma 2, we obtain for $n > m$ that

$$\mathbb{E} \left[\sup_{t \in [0,1]} \left| X_3^{(n)}(t) - X_3^{(m)}(t) \right|^2 \right] \leq 4 \int \mathbf{1}_{\{\epsilon_m > |x| \geq \epsilon_n\}} x^2 \Pi(dx).$$

This converges to 0 as $n, m \rightarrow \infty$ because $\int (x^2 \wedge 1) \Pi(dx) < \infty$, and hence there is a subsequence, denoted by (n') and a process $X_3(t)$ with almost surely càdlàg paths such that

$$\sup_{t \in [0,1]} \left| X_3^{(n')}(t) - X_3(t) \right| \rightarrow 0, \quad \text{a.s. when } n' \rightarrow \infty.$$

The limiting process must be an \mathbb{F} -martingale, as a convergence argument shows. Moreover

$$\begin{aligned} \mathbb{E} \left[e^{iuX_3(t)} \right] &= \lim_{n \rightarrow \infty} \mathbb{E} \left[e^{iu[Z_3^{(n)}(t) - t \int \mathbf{1}_{\{1 > |x| \geq \epsilon_n\}} x \Pi(dx)]} \right] \\ &= \lim_{n \rightarrow \infty} \exp \left\{ t \int_{\epsilon_n \leq |x| < 1} [e^{iux} - 1 - iux] \Pi(dx) \right\} \\ &= \exp \left\{ t \int_{(-1,1)} [e^{iux} - 1 - iux] \Pi(dx) \right\} \quad \diamond \end{aligned}$$