Filtrations, Stopping times, and some applications: Math 642:592, Spring 2008

1. Filtrations for continuous time processes.

A filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ in a measure space $(\Omega, \mathcal{F})$ is an increasing family of sub-σ-algebras of $\mathcal{F}$. Here t ranges through all non-negative real numbers. A *filtered probability space* is a quadruple $(\Omega, \mathcal{F}, I\!\!P, \mathbb{F})$ consisting of a probability space and a filtration in $(\Omega, \mathcal{F})$. A stochastic process X is adapted to a filtration $\mathbb{F}$ if $X(t)$ is $\mathbb{F}_t$-measurable for all t.

The following definition is very important in stochastic calculus, although it may appear mysterious at first. Some of the results we state in this lecture will begin to explain its purpose. In what follows

$$\mathcal{F}_{t+} \stackrel{\triangle}{=} \bigcap_{s>t} \mathcal{F}_s,$$

Given a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, let $\mathbb{F}_+$ denote the filtration $\{\mathcal{F}_{t+}\}$.

Definition. A filtered probability space $(\Omega, \mathcal{F}, I\!\!P, \mathbb{F})$ (or if the probability space is understood, the filtration $\mathcal{F}$) is said to satisfy the "usual conditions," if

(i) $(\Omega, \mathcal{F}, I\!\!P)$ is complete; recall this means that if $A \subset B$, $B \in \mathcal{F}$ and $P(B) = 0$, then $A \in \mathcal{F}$.

(ii) $\mathcal{F}_0$ contains all events of measure zero (and hence so does $\mathcal{F}_t$ for all $t > 0$.

(iii) $\mathbb{F}$ is right-continuous. This means that $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$.

Comment: If $\mathbb{F}$ is a given filtration, $\mathbb{F}_+$ is always right continuous.

In practice, filtrations often are generated from stochastic processes. Let X be a stochastic process on $(\Omega, \mathcal{F}, I\!\!P)$. Define the filtration $\overset{\circ}{\mathbb{F}}{}^X$ by

$$\overset{\circ}{\mathcal{F}}{}^X_t \stackrel{\triangle}{=} \sigma\{X(s)\,;\, 0 \leq s \leq t\}.$$

This is the "raw" filtration generated by X.

Let $\mathcal{N}$ be the collection of zero probability events in $(\Omega, \mathcal{F}, I\!\!P)$. The filtration obtained by augmenting the raw filtration generated by a process X with $\mathcal{N}$ will be called $\mathbb{F}^X$, and is defined by

$$\mathcal{F}^X_t \stackrel{\triangle}{=} \overset{\circ}{\mathcal{F}}{}^X_t \vee \mathcal{N}.$$

(Here $\overset{\circ}{\mathcal{F}}{}^X_t \vee \mathcal{N}$ denotes the smallest σ-algebra containing both $\overset{\circ}{\mathcal{F}}{}^X_t$ and $\mathcal{N}$.)

Theorem 1 *Let X be a Lévy process. Then $\mathbb{F}^X$ is right-continuous. Thus if X is a Lévy process on a complete probability space $(\Omega, \mathcal{F}, I\!\!P, \mathbb{F}^X)$ satisfies the usual conditions.*

Partial proof: Since $X(0) = 1$ with probability one, $\mathcal{F}_0^X$ is the σ-algebra consisting of sets of probability zero or of probability one. We will show that $\mathcal{F}_{0+}^X$ is trivial, which will imply right continuity at $t = 0$. This can be derived as a consequence of Kolmogorov's 0-1 law. But we will derive it from scratch here, using the method of proof of Kolmogorov's 0-1 law.

Observe that to prove $\mathcal{F}_{0+}^X$ is trivial it suffice to show that for any $A \in \mathcal{F}_{0+}^X$, $I\!\!P(A)^2 = I\!\!P(A)$, because then $I\!\!P(A) = 0$ or $I\!\!P(A) = 1$. This will be the goal. To achieve it, we will use the following, standard result of measure theory.

- If $\mathcal{C}$ is an algebra and if Q is a probability measure on $\sigma(\mathcal{C})$, then for every $A \in \sigma(\mathcal{C})$ and every $\epsilon > 0$, there is a $A_\epsilon \in \mathcal{C}$ such that $\mid Q(A) - Q(A_\epsilon) \mid \leq Q(A \triangle A_\epsilon) < \epsilon$. (This is proved easily by showing that the set of A such that for every $\epsilon > 0$ there is an $A_\epsilon \in \mathcal{C}$ with $Q(A \triangle A_\epsilon) < \epsilon$ is a monotone class containing $\mathcal{C}$.)

If $0 < s < t$, let $\mathcal{F}_{s,t}^X$ be the σ-algebra generated by the increments of X after s up to time t; that is $\mathcal{F}_{s,t}^X = \sigma\{X(u) - X(s)\,;\, s < u \leq t\} \vee \mathcal{N}$. By right continuity of X, for any $t > 0$,

$$\mathcal{F}_t^X \subset \sigma\left(\bigcup_{0<s<t} \mathcal{F}_{s,t}^X\right). \tag{1}$$

Also, since for fixed t, $\mathcal{F}_{s,t}^X$ is increasing as s decreases,

$$\bigcup_{s<t} \mathcal{F}_{s,t}^X \quad \text{is an algebra of sets.} \tag{2}$$

Let $A \in \mathcal{F}_{0+}$. We will show that $I\!\!P(A)^2 = I\!\!P(A)$ and hence that $I\!\!P(A)$ equals zero or one. Fix any $\epsilon > 0$ and any $t > 0$. Then, since $A \in \mathcal{F}_t$, there is an $0 < s < t$ and an $A_1 \in \mathcal{F}_{s,t}^X$ for which $I\!\!P(A \triangle A_1) < \epsilon$, by the principle we stated above, because of (1) and (2). By the same reasoning, there is an $0 < r < s$ and $A_2 \in \mathcal{F}_{r,s}^X$ such that $I\!\!P(A \triangle A_2) < \epsilon$. Since

$$A \triangle [A_1 \cap A_2] \subset [A \triangle A_1] \cup [A \triangle A_2],$$

it follows that

$$\Big| \, I\!\!P(A) - I\!\!P(A_1 \cap A_2) \, \Big| \leq I\!\!P\left(A \triangle [A_1 \cap A_2]\right) < 2\epsilon.$$

On the other hand, because X is an independent increment process, the σ-algebras $\mathcal{F}_{s,t}^X$ and $\mathcal{F}_{r,s}^X$ are independent, and hence $I\!\!P(A_1 \cap A_2) = I\!\!P(A_1) I\!\!P(A_2)$ and

$$\Big| \, I\!\!P^2(A) - I\!\!P(A_1 \cap A_2) \, \Big| \leq \Big| \, I\!\!P(A) - I\!\!P(A_1) \, \Big| + \Big| \, I\!\!P(A) - I\!\!P(A_2) \, \Big| < 2\epsilon.$$

It follows that $\Big| \, I\!\!P(A) - I\!\!P^2(A) \, \Big| < 4\epsilon$. Since ϵ was arbitrary, we conclude that $I\!\!P^2(A) = I\!\!P(A)$ and this completes the proof. $\diamond$

The proof for general $t > 0$ is similar, but requires a bit more fussing. We will sketch it here as a series of exercises.

Exercise 1. (a) For any $s > t$, $\mathcal{F}_s^X = \sigma\left(\mathcal{F}_t^X \vee \bigcup_{t<r<s} \mathcal{F}_{r,s}^X\right)$.

(b) Let $\mathcal{A}$, $\mathcal{G}$, $\mathcal{H}$ be mutually independent σ-algebras. Then if $A \in \mathcal{A} \vee \mathcal{G}$ and if $B \in \mathcal{A} \vee \mathcal{H}$, then

$$\mathbb{P}(A \cap B \mid \mathcal{A}) = \mathbb{P}(A \mid \mathcal{A})\mathbb{P}(B \mid \mathcal{A}). \tag{3}$$

Hints: A simple calculation shows that (3) is true for $A = U \cap G$ and $B = V \cap H$, where $U \in \mathcal{A}$, $V \in \mathcal{A}$, $G \in \mathcal{G}$, and $H \in \mathcal{H}$. For fixed $A \in \mathcal{A} \vee \mathcal{G}$ show that $\mathcal{D}_A = \{B \in \mathcal{A} \vee \mathcal{H}\,;\,(3) \text{ holds}\}$ is a d-system (see RW,page 87). Note that $\mathcal{A} \vee \mathcal{G}$ is generated by sets of the form $U \cap G$, where $U \in \mathcal{A}$ and $G \in \mathcal{G}$, and $\mathcal{A} \vee \mathcal{H}$ is generated by sets of the form $V \cap H$, where $V \in \mathcal{A}$ and $H \in \mathcal{H}$. Then, for a set of the form $A = U \cap G$, use the $\pi-$, $d-$system theorem (see Proposition I.7, page 87) to show that $\mathcal{D}_A$ contains $\mathcal{A} \vee \mathcal{H}$. Now fix any $B \in \mathcal{A} \vee \mathcal{H}$ and use the same method to show that (3) is true for any $A \in \mathcal{A} \vee \mathcal{G}$.

(c). Show that if $\mathbb{P}(A \mid \mathcal{F}_t^X) = (\mathbb{P}(A \mid \mathcal{F}_t^X))^2$, a.s. for all $A \in \mathcal{F}_{t+}^X$ then $\mathcal{F}_{t+}^X = \mathcal{F}_t^X$.

Hint: Let $A' = \{\omega\,;\,\mathbb{P}(A \mid \mathcal{F}_t^X = 1\}$ and note that $\mathbf{1}_{A'} = \mathbb{P}(A \mid \mathcal{F}_t^X)$ almost surely. Then show that $\mathbb{E}\left[(\mathbf{1}_A - \mathbf{1}_{A'})^2\right] = 0$.

(d). Fix $A \in \mathcal{F}_{t+}^X$. Establish the existence of a sequence $\{A_n^{(1)}, A_n^{(2)}, t < r_n < s_n < t_n\}$, such that $A_n^{(1)} \in \mathcal{F}_t^X \vee \mathcal{F}_{s_n,t_n}^X$, $A_n^{(2)} \in \mathcal{F}_t^X \vee \mathcal{F}_{r_n,s_n}^X$, for all $n \geq 1$, $t_n \downarrow t$, and $\lim_{n\to\infty} \mathbb{P}(A \triangle A_n^{(i)}) = 0$, $i = 1, 2$. Use the result of (b) to show that $\mathbb{P}(A \mid \mathcal{F}_t^X) = (\mathbb{P}(A \mid \mathcal{F}_t^X))^2$. It is helpful to observe that if $\mathbb{P}(B \triangle B_n) \to 0$ as $n \to \infty$, then

$$\mathbb{E}\left[\left|\, \mathbb{P}(B \mid \mathcal{G}) - \mathbb{P}(B_n \mid \mathcal{G}) \,\right|\right] \leq \mathbb{E}\left[\left|\, \mathbf{1}_B - \mathbf{1}_{B_n} \,\right|\right] = \mathbb{P}(B \triangle B_n) \to 0$$

as $n \to \infty$ and so there exists a subsequence $\{n_k\}$ for which $\mathbb{P}(B_{n_k} \mid \mathcal{G})$ converges a.s. to $\mathbb{P}(B \mid \mathcal{G})$ as $k \to \infty$.

We can use this 0-1 law to derive a simple but important fact about Brownian motion.

Corollary 1 *Let W be a Brownian motion and let $T^+ = \inf\{t > 0\,;\, W(t) > 0\}$ and $T^- = \inf\{t > 0\,;\, W(t) < 0\}$. Then $\mathbb{P}(T^+ = 0, T^- = 0) = 1$.*

Proof. If W is a Brownian motion, then so is $-W$ so it suffices to show that $\mathbb{P}(T^+ = 0) = 1$. The event $\{T == 0\}\mathcal{F}_{0+}$, because it can be written as a decreasing intersection

$$\{T^+ = 0\} = \bigcap_{n\geq 1} \{\exists\, s \in \mathbb{Q} \cap [0, 1/n]\,, W(s) > 0\}\,.$$

and $\{\exists\, s \in \mathbb{Q} \cap [0, 1/n]\, W(s) > 0\} \in \mathcal{F}_{1/n}^W$ for all $n \geq 1$. Therefore $\{T^+ = 0\}$ has probability equal to either 0 or 1. However, for every $n \geq 1$,

$$\mathbb{P}\left(\exists\, s \in \mathbb{Q} \cap [0, 1/n]\,, W(s) > 0\right) \geq \mathbb{P}\left(W(1/n) > 0\right) = 1/2.$$

By continuity from above of probability measures, it follows that $\mathbb{P}(\{T^+ = 0\} \geq 1/2$, and hence it must have probability 1. $\diamond$

Suppose we are given a filtered probability space $(\Omega, \overset{\circ}{\mathcal{F}}, \overset{\circ}{I\!\!P}, \mathbb{F})$. Let us first construct the $I\!\!P$-completion, $(\omega, \mathcal{F}, I\!\!P)$, of $(\Omega, \overset{\circ}{\mathcal{F}}, I\!\!P)$. We know how to do this from Lecture Note 1, RW, pp. 94-95. Let $\mathcal{N}$ be the zero probability events of this completed probability space. Define now the filtration $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t\}_{t \geq 0}$, by

$$\bar{\mathcal{F}}_t = \bigcap_{s>t} \mathcal{F} \vee \mathcal{N}.$$

This filtration satisfies the usual conditions and is called the *usual augmentation* of $\mathbb{F}$.

Theorem 1 showed that we can form the usual augmentation of to the filtration associated to a Lévy process and obtain a filtration satisfying the usual conditions. Similarly we can usually augment the filtrations of martingales without harm. Let a filtered probability space be given with filtration $\mathbb{F}$. An $\mathbb{F}$-martingale (submartingale, supermartingale) is an $\mathbb{F}$-adapted, integrable process that satisfies

$$\mathbb{E}\left[X(t) \mid \mathcal{F}_s\right] = X(s) \quad \text{(respectively } \geq, \leq \text{) a.s. for all } 0 \leq s < t < \infty.$$

Lemma 1 . *Let M be an $\mathbb{F}$-submartingale with a.s. right-continuous paths on a filtered probability space with filtration $\mathbb{F}$. Then M is an $\bar{\mathbb{F}}$-submartingale, where $\bar{\mathbb{F}}$ is the usual augmentation of $\mathbb{F}$.*

Proof. Without loss of generality we may assume that the underlying probability space is already complete. M will certainly be a martingale with respect to the filtration $\{\mathcal{F}_t \vee \mathcal{N}; t \geq 0\}$ because the events of $\mathcal{F}_t$ and $\mathcal{F}_t \vee \mathcal{N}$ differ by zero probability events. Fix $0 \leq s < t$ and let $\{s_n\}$ be a sequence such that $t > s_1 > s_2 > \cdots s$ and $\lim_{n \to \infty} s_n = s$. Then

$$M(s) = \lim_{n \to \infty} M(s_n).$$

However for every n, $M(s_n) \leq \mathbb{E}\left[M(t) \mid \mathcal{F}_{s_n} \vee \mathcal{N}\right]$. By Lévy's downward theorem,

$$\lim_{n \to \infty} \mathbb{E}\left[M(t) \mid \mathcal{F}_{s_n} \vee \mathcal{N}\right] = \mathbb{E}\left[M(t) \,\middle|\, \cap_{s>t}(\mathcal{F}_s \vee \mathcal{N})\right] = \mathbb{E}\left[M(t) \mid \bar{\mathcal{F}}_t\right] \quad \text{a.s.}$$

It follows that $M(s) \leq \mathbb{E}\left[M(t) \mid \bar{\mathcal{F}}_t\right]$, a.s. Since this is true for any $0 \leq s < t$, it follows that M is an $\bar{\mathbb{F}}$-martingale. ◇

This raises the deeper question: in what generality martingales and submartingales have right-continuous or càdlàg paths. It is important to have an answer, because path regularity is required to extend martingale inequalities and optional stopping to continuous time. Fortunately, Doob's theorems for discrete time help to establish such regularity. Here is the main result, given without proof. (See RW, pp. 169-174.)

Theorem 2 *Let Y be an $\mathbb{F}$-sub martingale on a filtered probability space with filtration $\mathbb{F}$. Assume that*

$$\lim_{s \downarrow t} \mathbb{E}\left[\mid Y_s - Y_t \mid\right] = 0, \quad \textit{for all } t \geq 0.$$

Then there is a càdlàg version X of Y (i.e. $I\!\!P(X(t) = Y(t)) = 1$ for all $t \geq 0$) such that X is an $\bar{\mathbb{F}}$-submartingale.

2. Stopping times and measurability.

Given a filtration $\mathbb{F}$, a random variable $T : \Omega \to [0,\infty]$ is an $\mathbb{F}$-stopping time if

$$\{T \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0. \tag{4}$$

A simple argument (exercise) shows that T is an $\mathbb{F}_+$-martingale if and only if

$$\{T < t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0. \tag{5}$$

Thus, when $\mathbb{F}$ is right-continuous, we can check whether T is a stopping time with respect to $\mathbb{F}$ by using (4) or (5).

The following lemma gives a useful example. If X is process, we define

$$H_a = \inf\{t;\, X(t) \in [a,\infty)\} \quad \text{and} \quad \widetilde{H}_a = \inf\{t;\, X(t) > a\}.$$

Lemma 2 *Let X be a process with continuous paths. Then H_a is an $\overset{\circ}{\mathbb{F}}{}^X$-stopping time. If X is right-continuous, $\widetilde{H}_a$ is an $\overset{\circ}{\mathbb{F}}{}^X{}_+$-stopping time.*

Proof: Let $\mathbb{Q}$ denote the rationals. If X has continuous paths,

$$\{H_a \leq t\} = \bigcap_n \bigcup_{s \in \mathbb{Q} \cap [0,t]} \{X(s) \in (a - 1/n, \infty)\}\,.$$

Since $\{X(s) \in (a-1/n,\infty)\} \in \mathcal{F}_t^X$ whenever $s \leq t$, it follows that $\{H_a \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$, proving that H_a is an $\overset{\circ}{\mathcal{F}}{}_t^X$-stopping time.

If X is right-continuous,

$$\{\widetilde{H}_a < t\} = \bigcup_{s \in \mathbb{Q} \cap [0,t)} \{X(s) \in (a,\infty)\}$$

and this is in $\mathcal{F}_t^X$ since every set in the union on the right-hand side belongs to $\mathcal{F}_t^X$. $\diamond$

Here is nice application of Theorem 1 and Corollary 1. We use without proof the fact that for a Brownian motion $H_a < \infty$ with probability one.

Corollary 2 *If W is a Brownian motion and $a \geq 0$, then $\mathbb{P}(H_a = \tilde{H}_a) = 1$.*

Proof. Just observe that

$$\{H_a = \tilde{H}_a\} = \Big\{\inf\{s\,;\, W^{H_a}(s) > 0\} = 0\Big\}\,.$$

The event on the right has probability 1, since W^{H_a} is a Brownian motion. $\diamond$

In general $\tilde{H}_a$ is not an $\overset{\circ}{\mathcal{F}}{}_t^X$-stopping time. Since first entrance times into sets are important in applications, it is important to know under what circumstances and with respect to what filtrations they are stopping times. This will be dealt with shortly, but first, in preparation, it is necessary to deal with some measurability issues.

In the discrete-time setting, it was no problem to talk about about $X(T(\omega))(\omega)$ where X is a discrete-time process and T is an integer-valued random variable. However, in continuous time, one must impose conditions in order that $\omega \to X(T(\omega))(\omega)$ be a valid random variable. A sufficient condition is joint measurability. A stochastic process X on $(\Omega, \mathcal{F}, I\!\!P)$ is called *measurable* if the map

$$(t, \omega) \to X(t)(\omega)$$

is measurable as a function on $[0, \infty)$ endowed with the product σ-algebra $\mathcal{B}([0, \infty)) \times \mathcal{F}$. Recall that this is product σ-algebra is the smallest σ-algebra containing all "rectangles" of the form $G \times U$, where G is a Borel subset of $[0, \infty)$ and $U \in \mathcal{F}$. When we wish to emphasize studying $X(t)(\omega)$ as a function of both t and ω, we shall sometimes write $X(t, \omega)$ instead of $X(t)(\omega)$.

Measurability is a very important property to have for even elementary operations. For example, even to be able to treat such relatively innocent object as integrals $\int_a^b X(s, \omega)\, \mathrm{d}s$ as random variables, requires measurability. Measurability also addresses the question immediately at hand.

Lemma 3 *Suppose that T is a random variable taking values in $[0, \infty)$. Let X be a measurable process. Then $X(T)$ is a random variable, that is, $\omega \to X(T(\omega), \omega)$ is a measurable function on $(\Omega, \mathcal{F}, I\!\!P)$.*

Proof: Write $X(T(\omega), \omega) = X \circ \Phi(\omega)$, where $\Phi : \Omega \to [0, \infty) \times \Omega$ is $\Phi(\omega) = (T(\omega), \omega)$. Observe that Φ is measurable as a map from $(\Omega, \mathcal{F})$ to $([0, \infty) \times \Omega, \mathcal{B}([0, \infty)) \times \mathcal{F})$. If X is measurable, $X(T)$ is the composition of two measurable maps and hence is measurable. ◇

From now on in this course, all processes are assumed to be measurable. The following result assures us that most of the processes we will deal with are measurable automatically.

Theorem 3 *If X is a right-continuous or left continuous process, then X is measurable.*

Proof: Consider the right-continuous case. Then for every (t, ω),

$$X(t, \omega) = \lim n \to \infty \sum_k X((k+1)/n)(\omega) \mathbf{1}_{[k/n, (k+1)/n)}(t).$$

and for each n, the sum on the right is clearly jointly measurable as a function of (t, ω). ◇

When X is measurable and is adapted to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ and T is an $\mathbb{F}$ stopping time, it is natural to want that $X(T)$ be $\mathcal{F}_T$-measurable also. However, this need not be the case and we need a strengthening of the notion of adaptedness. We say that a process is $\mathbb{F}$-progressively measurable if for each $\tau > 0$, the function $(t, \omega) \to X(t, \omega)$ restricted to $(t, \omega) \in [0, \tau] \times \Omega$ is measurable as a map on $([0, \tau] \times \Omega, \mathcal{B}([0, \tau]) \times \mathcal{F})$. It is clear from the definition that an $\mathbb{F}$-progressively measurable process is both measurable and adapted.

Theorem 4 *Let X be $\mathbb{F}$-progressively measurable and let T be an $\mathbb{F}$-stopping time. Then $X(T)$ is $\mathbb{F}_T$-measurable.*

Proof: We need to show that for any Borel U and any $t \geq 0$, $\{X(T) \in U\} \cap \{T \leq t\} \in sF_t$, or equivalently, that $X(T(\omega), \omega)$ is $\mathcal{F}_t \cap \{T \leq t\}$ measurable when restricted to $\{T \leq t\}$. The proof is similar in spirit to the proof of Lemma 3. For each t, $X(T(\omega), \omega)$ restricted to $\{T \leq t\}$ is the composition of X restricted to $[0, t] \times \{T \leq t\}$ with $\Phi(\omega) = (\omega, T(\omega)$ restricted to $\{T \leq t\}$. Since X is measurable with respect to $\mathcal{B}([0, t]) \times \mathcal{F}_t$ as a map on $[0, t] \times \Omega$ and $\{T \leq t\} \in \mathcal{F}_t$, X is measurable with respect to $\mathcal{B}([0, t]) \times [\{T \leq t\} \cap \mathcal{F}_t]$ when restricted to $[0, t] \times \{T \leq t\}$. Φ restricted to $\{T \leq t\}$ is measurable as a map from $(\{T \leq t\}, \{T \leq t\} \cap \mathcal{F}_t)$ to $\mathcal{B}([0, t]) \times [\{T \leq t\} \cap \mathcal{F}_t]$. Hence the composite map $X(T(\omega), \omega)$ is $\mathcal{F}_t$ measurable when restricted to $\{T \leq t\}$. $\diamond$

Theorem 5 *Let X be $\mathbb{F}$ adapted and suppose either that X is right-continuous or left-continuous. Then X is $\mathbb{F}$-progressively measurable.*

The proof is virtually the same as that of Theorem 3. Consider the case in which X is right-continuous. Then for every n,

$$(s, \omega) \to X(\tau)\mathbf{1}_{s=\tau} + \sum_{k=1}^{n} X(k\tau/n)\mathbf{1}_{[(k-1)\tau/n, k\tau)}(s)$$

is measurable with respect to $(\mathcal{B}([0, \tau]) \times \mathcal{F}$ and $X(s, \omega)$ is a point-wise limit of these processes as $n \to \infty$.

The following, very difficult theorem, called the *Début Theorem*, gives an answer to the question of when hitting times are stopping times. It generalizes to processes with values in a metric space; see RW, p. 186.

Theorem 6 *Let $(\Omega, \mathcal{F}, I\!\!P, \mathbb{F})$ be a filtered probability space. Let X be $\mathbb{F}$-progressively measurable. Let U be a Borel subset of $\mathbb{R}$. Then $T_B \overset{\triangle}{=} \inf\{T\,;\, X(t) \in B\}$ is a stopping time with respect to the usual augmentation $\bar{\mathbb{F}}$ of $\mathbb{F}$.*

3. The strong Markov property for Lévy processes.

In this section X is a Lévy process with respect to a filtration $\mathbb{F}$. Recall that this means that besides being a Lévy process, X is adapted to $\mathbb{F}$ and for each $t \geq 0$, the process of forward increments $\{X(t+s) - X(t)\,;\, s \geq 0\}$ is independent of $\mathcal{F}_t$. If we complete the underlying probability space and augment the σ-algebras of the filtration with the zero probability sets X will still be a Lévy process with respect to this augmented filtration. So we can assume without loss of generality that this augmentation has been made so that the underlying probability space is complete and $\mathcal{F}_0$ contains all the zero probability events. Then $\mathbb{F}_+$ is the usual augmentation of $\mathbb{F}$. As the paths of X are right-continuous, we know that X is $\mathbb{F}$-progressively measurable.

If T is an $\mathbb{F}_+$ stopping time such that $T(\omega) < \infty$ for all ω, we define

$$X^T(t) = X(T+t) - X(T), \quad t \geq 0.$$

The following theorem extends the independent increments property to stopping times and establishes the strong Markov property of Lévy processes.

Theorem 7 *Let X be a Lévy processes with respect to $\mathbb{F}$. If T is an $\mathbb{F}_+$ stopping time such that $T(\omega) < \infty$, then X^T is a Lévy process with the same finite dimensional distributions (and hence the same Lévy-Khinchine representation) as X, and X^T is independent of $\mathcal{F}_{T+}$.*

Proof: We begin with a preliminary tool for simplifying the proof of independence. Recall that to say a process Y is independent of a σ-algebra $\mathcal{G}$ means that $\sigma(\{Y(s)\,;\, s \geq 0\})$ is independent of $\mathcal{G}$. To establish this fact it suffices to show

$$\begin{gathered}\mathbb{E}\left[\mathbf{1}_A \mathbf{1}_U\left(Y(s_1), \ldots, Y(s_m)\right]\right. = \mathbb{E}\left[\mathbf{1}_A\right] \mathbb{E}\left[\mathbf{1}_U\left(Y(s_1), \ldots, Y(s_m)\right)\right] \\ \text{for every } A \in \mathcal{G},\ m \geq 0,\ 0 \leq s_1 < \cdots < s_m,\ U \in \mathcal{B}(\mathbb{R}^m).\end{gathered} \tag{6}$$

And to show this, it suffices to show that

$$\begin{gathered}\mathbb{E}\left[\mathbf{1}_A h(Y(s_1), \ldots, Y(s_m)\right] = \mathbb{E}\left[\mathbf{1}_A\right] \mathbb{E}\left[h(Y(s_1), \ldots, Y(s_m)\right] \\ \text{for every } A \in \mathcal{G},\ m \geq 0,\ 0 \leq s_1 < \cdots < s_m,\ h \text{ bounded and continuous.}\end{gathered} \tag{7}$$

The reason that (6) is sufficient is that it shows that every event in $\mathcal{G}$ is independent of every event in the algebra

$$\bigcup_{m \geq 1, s_1, \ldots, s_m \geq 0} \sigma\left\{Y(s_1), \ldots, Y(S_m)\right\}.$$

By the monotone class theorem, this is enough to show that $\mathcal{G}$ is independent of the σ-algebra generated by Y. To show that (7) is sufficient for independence, observe that for any open U in $\mathbb{R}^n$, $\mathbf{1}_U$ may be realized as the limit of a sequence of bounded, continuous functions. Thus, (7) implies (6) for U restricted to be open. But for a fixed A, the set of U for which (6) holds is a d-system and so the π-, d-system theorem imples (6) holds for all Borel U.

We shall also use a second preliminary result on stopping times.

Lemma 4 *(a) Let T be an $\mathbb{F}_+$-stopping time and let S be an $\mathbb{F}$-stopping time such that $T < S$ on $\{T < \infty\}$. Then $\mathcal{F}_{T+} \subset \mathcal{F}_S$.*

(b) Let T be an $\mathcal{F}_+$-stopping time. Let $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_{n\to\infty} t_n = \infty$. Then $S \overset{\triangle}{=} \sum_{k=1}^{\infty} t_k \mathbf{1}_{[t_{k-1}, t_k)}(T)$ is an $\mathbb{F}$-stopping time and $S > T$ on $\{T < \infty\}$.

Proof. (a) Let $A \in \mathcal{F}_{T+}$. Then for any $t \geq 0$, $A \cap \{S \leq t\} = A \cap \{T < t\} \cap \{S \leq t\}$. But $A \cap \{T < t\}$, by the definition of $\mathcal{F}_{T+}$ and $\{S \leq t\} \in \mathcal{F}_t$ because S is an $\mathbb{F}$ stopping time. Therefore $A \cap \{S \leq t\}$. Thus $A \in \mathcal{F}_S$.

(b) To show that S is a stopping time, it is only necessary to show $\{S = t_k\} \in \mathcal{F}_{t_k}$ for all k. But $\{S = t_k\} = \{t_{k-1} \leq T < t_k\}$, and this is in $\mathcal{F}_{t_k}$ because T is an $\mathcal{F}_{T+}$-stopping time. ◇

We return to the proof of Theorem 7. First we shall prove that if T is an $\mathbb{F}$-stopping time that takes values in a countable set $\{t_k;\, k \geq 1\}$, then X^T is a Lévy process that is independent $\mathcal{F}_T$. Indeed, let $A \in \mathcal{F}_T$. Then for every k, $A \subset \{T = t_k\} \in \mathcal{F}_{t_k}$. By the independence and stationarity of the increments of X, $(X(t_k+s_1) - X(t_k), \ldots, X(t_k+s_m) - X(t_k))$ is independent of $\mathcal{F}_T$ and has the same distribution as $(X(s_1), \ldots, X(s_m))$. Thus, if U is a Borel subset of $\mathbb{R}^m$,

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_A \mathbf{1}_U\left(X^T(s_1), \ldots, X^T(s_m)\right)\right] &= \sum_k \mathbb{E}\left[\mathbf{1}_{A\cap\{T=t_k\}} \mathbf{1}_U\left(X(t_k+s_1) - X(t_k), \ldots, X(t_k+s_m) - X(t_k)\right)\right] \\
&= \sum_k \mathbb{E}\left[\mathbf{1}_{A\cap\{T=t_k\}}\right] \mathbb{E}\left[\mathbf{1}_U\left(X(s_1), \ldots, X(s_m)\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_A\right] \mathbb{E}\left[\mathbf{1}_U\left(X(s_1), \ldots, X(s_m)\right)\right].
\end{aligned}$$

By the criterion of (7), this proves the desired independence. By taking $A = \Omega$, it follows that

$$\mathbb{E}\left[\mathbf{1}_U\left(X^T(s_1), \ldots, X^T(s_m)\right)\right] = \mathbb{E}\left[\mathbf{1}_U\left(X(s_1), \ldots, X(s_m)\right)\right]$$

for any m, $0 \leq s_1 < s_2 < \cdots < s_m$, and $U \in sB(I\!\!R^m)$, and this shows the equality of the finite dimensional distributions of X^T and X.

Now consider a general, finite-valued, $\mathbb{F}_+$ stopping time T. For any $n \geq 1$, define

$$T^n = \sum_{k=1}^{\infty} \frac{k}{n} \mathbf{1}_{[(k-1)/n, k/n)}(T).$$

Let $A \in \mathcal{F}_T+$. By Lemma 4, $A \in \mathcal{F}_{T^n}$. Therefore, by what we have proved for countably-valued stopping times, for any $m \geq 1$, $0 \leq s_1 < \cdots < s_m$ and any bounded, continuous h on $\mathbb{R}^n$,

$$\mathbb{E}\left[\mathbf{1}_A h\left(X^{T^n}(s_1), \ldots, X^{T^n}(s_m)\right)\right] = \mathbb{E}\left[\mathbf{1}_A\right] \mathbb{E}\left[h\left(X(s_1), \ldots, X(s_m)\right)\right].$$

By right-continuity of the process X, we therefore obtain

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_A h\left(X^T(s_1), \ldots, X^T(s_m)\right)\right] &= \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_A h\left(X^{T^n}(s_1), \ldots, X^{T^n}(s_m)\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_A\right] \mathbb{E}\left[h\left(X(s_1), \ldots, X(s_m)\right)\right].
\end{aligned}$$

By the criterion of (7), this proves the independence of X^T and $\mathcal{F}_{T+}$. By setting $A = \omega$, we obtain as before the identity in law of X and X^T. ◇

4. The reflection principle for Brownian motion.

Brownian motion is invariant (in law) under reflection. That is, if W is a Brownian motion, so also is $-W$. Even more, Brownian motion is invariant reflection of its increments past any time. This means that for any fixed, deterministic r,

$$\widetilde{W}(t) \triangleq \begin{cases} W(t), & 0 \leq t \leq r; \\ W(r) - [W(t) - W(r)], & t > r. \end{cases}$$

is again a Brownian motion. The next result, called the reflection principle generalizes this fact to any a.s. finite stopping time.

Theorem 8 *Let W be an $\mathbb{F}$-Brownian motion. Let T be a $\mathbb{F}_{T+}$ stopping time. Then*

$$\widetilde{W}(t) \triangleq \begin{cases} W(t), & 0 \le t \le T; \\ W(T) - [W(t) - W(T)], & t > r. \end{cases}$$

is an $\mathbb{F}$-Brownian motion.

Proof. There is a slick way to prove this, but it takes a little bit of measure theoretic preparation. We will sketch the idea without doing all the details and then present a more pedestrian proof. Let $C_0[0,\infty)$ be the space of continuous functions starting at 0. Define a map $\Psi : [0,\infty) \times C_0[0,\infty) \times C_0[0,\infty) \to C_0[0,\infty)$ as follows

$$\Psi(r,\eta,\rho)(t) = \begin{cases} \eta(t), & \text{if } t \le r; \\ \eta(r) + \rho(t-r), & \text{if } t > r. \end{cases}$$

Let W_T denote the Brownian motion stopped at T: that is, $W_T(t) \overset{\triangle}{=} W(t \wedge t)$. Clearly, $W(t) = \Psi(T, W_T, W^T)(t)$. But by the strong Markov property of Theorem 7, W^T is a Brownian motion independent of (T, W_T). Therefore, $\Psi(T, W_T, B)(t)$, $t \ge 0$, defines a Brownian motion for any Brownian motion B independent of (T, W_T). In particular, $\Psi(T, W_T, -W^T)(t)$, $t \ge 0$, defines a Brownian motion. However, a simple check of the definition shows that $\Psi(T, W_T, -W^T)$ is the same as $\widetilde{W}$.

Now for the more pedestrian proof. As for the proof of Theorem 7, let us first assume that T takes values in a countable set $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$ and that T is an $\mathbb{F}$-stopping time. Fix any $r \ge 0$ and $A \in \mathcal{F}_r$. Let $r = s_0 < s_1 < s_2 < \cdots s_m$. We let $\triangle_i \widetilde{W} = W(s_i) - W(s_{i-1})$ and we define $\triangle_i \widetilde{W}$ similarly. Note that if $T \le s_{i-1} < s_i$, we have that $\triangle_i \widetilde{W} = -\triangle_i W$. We will be done if we can show that for any m and $r = s_0 < s_1 < s_2 < \cdots s_m$, and any bounded, Borel function h on $\mathbb{R}^m$,

$$\mathbb{E}\left[\mathbf{1}_A h\left(\triangle_1 \widetilde{W}, \ldots, \triangle_m \widetilde{W}\right)\right] = \mathbb{E}\left[\mathbf{1}_A\right] \mathbb{E}\left[h\left(\triangle_1 W, \ldots, \triangle_m W\right)\right] \tag{8}$$

We may assume without loss of generality that the set $\{s_0, s_1, \ldots, s_m\}$ contains each value of t_k such that $r = s_0 \le t_k < s_m\}$. Suppose for example that $s_{i-1} < t_k < s_i$. Then, since

$$\triangle_i \widetilde{W} = \widetilde{W}(s_i) - \widetilde{W}(s_{i-1}) = [\widetilde{W}(s_i) - \widetilde{W}(t_k)] + [\widetilde{W}(t_k) - \widetilde{W}(s_{i-1})],$$

we can write

$$h\left(\triangle_1 \widetilde{W}, \ldots, \triangle_m \widetilde{W}\right) = h\left(\triangle_1 \widetilde{W}, \ldots, [\widetilde{W}(s_i) - \widetilde{W}(t_k)] + [\widetilde{W}(t_k) - \widetilde{W}(s_{i-1})], \ldots, \triangle_m \widetilde{W}\right),$$

which is a bounded Borel function of the increments $(\triangle_1 \widetilde{W}, \ldots, [\widetilde{W}(s_i) - \widetilde{W}(t_k)] + [\widetilde{W}(t_k) - \widetilde{W}(s_{i-1})], \ldots, \triangle_m \widetilde{W})$.

We claim now that for any k

$$\mathbb{E}\left[\mathbf{1}_{A\cap\{T=t_k\}}h\left(\triangle_1\widetilde{W},\ldots,\triangle_m\widetilde{W}\right)\right]=\mathbb{E}\left[\mathbf{1}_{A\cap\{T=t_k\}}h\left(\triangle_1 W,\ldots,\triangle_m W\right)\right] \tag{9}$$

There are three cases to consider. First suppose that $t_k\le r$. On $\{T=t_k\}$,

$$\left(\triangle_1\widetilde{W},\ldots,\triangle_m\widetilde{W}\right)=-\left(\triangle_1 W,\ldots,\triangle_m W\right)$$

and moreover this is independent of $A\cap\{T=t_k\}\in\mathcal{F}_{t_k}$ and has the same distribution as

$$\left(\triangle_1 W,\ldots,\triangle_m W\right).$$

Therefore (9) follows immediately for this case.

The second case is when $t_k\ge s_m$. Then, on $\{T=t_k\}$,

$$\left(\triangle_1\widetilde{W},\ldots,\triangle_m\widetilde{W}\right)=\left(\triangle_1 W,\ldots,\triangle_m W\right)$$

and (9) follows even more easily.

The last case is when $t_k=s_j$ for some $1\le j\le m-1$. To save ourselves messy general notation, consider just the case $t_k=s_1$. Then

$$\left(\triangle_1\widetilde{W},\ldots,\triangle_m\widetilde{W}\right)=\left(\triangle_1 W,-\triangle_2 W\ldots,-\triangle_m W\right).$$

But by the independence of increments and the fact that the law of each increment and its negative are the same,

$$\mathbb{E}\left[h\left(\triangle_1 W,-\triangle_2 W\ldots,-\triangle_m W\right)\,\middle|\,\mathcal{F}_{s_1}\right]=\mathbb{E}\left[h\left(\triangle_1 W,\triangle_2 W\ldots,\triangle_m W\right)\,\middle|\,\mathcal{F}_{s_1}\right]$$

. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{A\cap\{T=s_1\}}h\left(\triangle_1\widetilde{W},\ldots,\triangle_m\widetilde{W}\right)\right]\\
&=\mathbb{E}\left[\mathbf{1}_{A\cap\{T=s_1\}}\mathbb{E}\left[h\left(\triangle_1 W,-\triangle_2 W\ldots,-\triangle_m W\right)\,\middle|\,\mathcal{F}_{s_1}\right]\right]\\
&=\mathbb{E}\left[\mathbf{1}_{A\cap\{T=s_1\}}\mathbb{E}\left[h\left(\triangle_1 W,\triangle_2 W\ldots,\triangle_m W\right)\,\middle|\,\mathcal{F}_{s_1}\right]\right]\\
&=\mathbb{E}\left[\mathbf{1}_{A\cap\{T=s_1\}}h\left(\triangle_1 W,\triangle_2 W\ldots,\triangle_m W\right)\right]
\end{aligned}$$

This completes the proof of (9). By summing on k, it follows that

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_A h\left(\triangle_1\widetilde{W},\ldots,\triangle_m\widetilde{W}\right)\right]&=\mathbb{E}\left[\mathbf{1}_A h\left(\triangle_1 W,\triangle_2 W\ldots,\triangle_m W\right)\right]\\
&=\mathbb{E}\left[\mathbf{1}_A h\right]\mathbb{E}\left[\left(\triangle_1 W,\triangle_2 W\ldots,\triangle_m W\right)\right]
\end{aligned}$$

The last equality follows because the future increments of W are independent of the past. This completes the proof for countably valued stopping times. The general case follows by approximating a general stopping time and taking limits, following the method of the proof of Theorem 7.

The standard first application of the reflection principle is deriving the distribution of H_a, the first time to hit level a, of a standard Brownian motion.

Theorem 9 *For a standard Brownian motion,*

$$\mathbb{P}(\tilde{H}_a \leq t) = \mathbb{P}(H_a \leq t) = 2\mathbb{P}(W(t) > a) = 2\int_{a/\sqrt{t}}^{\infty} e^{-s^2/2} \frac{\mathrm{d}s}{\sqrt{2\pi}}. \tag{10}$$

Proof: Corollary 2 implies that $\mathbb{P}(\tilde{H}_a > t) = \mathbb{P}(H_a > t)$.

Notice that H_a is also the first time that

$$\widetilde{W}(t) \triangleq \begin{cases} W(t), & 0 \leq t \leq H_a; \\ W(H_a) - [W(t) - W(H_a)], & t > r \end{cases}$$

hits level a. Also notice that when $H_a \leq t$ and $W(t) < a$, $\widetilde{W}(t) > a$. Thus,

$$\begin{aligned} \mathbb{P}(H_a \leq t) &= \mathbb{P}(H_a \leq t, W(t) \geq a) + \mathbb{P}(H_a \leq t, W(t) < a) \\ &= \mathbb{P}(W(t) \geq a) + \mathbb{P}(H_a \leq t, \widetilde{W}(t) > a) \\ &= \mathbb{P}(W(t) \geq a) + \mathbb{P}(\widetilde{W}(t) > a) = 2\mathbb{P}(W(t) > a). \qquad \diamond \end{aligned}$$

5. A study of the process $\{\tilde{H}_a\,;\, a \geq 0\}$.

Let W be a standard Brownian motion. In this section, we study $\{\tilde{H}_a \;\; a \geq 0\}$, where

$$\tilde{H}_a = \inf\{t \geq 0\,;\, W(t) > a\}$$

as a stochastic process indexed by $a \geq 0$. We will show that it is a non-decreasing Lévy process increasing by jumps only, and we will identify its characteristic measure.

It is obvious that $\tilde{H}_a$ is non-decreasing. It is also not hard to see that it is right-continuous. By the path-continuity of W, $W(\tilde{H}_a) = a$. Suppose $a_n \downarrow a$, where $a_n > a$ for all $n \geq 1$. For every $\epsilon > 0$ and every ω there is a t with $0 < t < \epsilon$, such that $W(\tilde{H}_a(\omega) + t)(\omega) > a$, by definition of $\tilde{H}_a$. Thus there is some n (also depending on ω) such that $W(\tilde{H}_a + t) > a_n > a$ implying that $\tilde{H}_{a_m}(\omega) < \tilde{H}_a(\omega) + \epsilon$ for all $m \geq n$. Since ϵ and ω were arbitrary, it follows that $\lim_{n\to\infty} \tilde{H}_{a_n}(\omega) \leq \tilde{H}_a(\omega)$ for every ω. But $\lim_{n\to\infty} \tilde{H}_{a_n} \geq \tilde{H}_a$ also, because $\tilde{H}_a$ is non-decreasing. Thus $\lim_{n\to\infty} \tilde{H}_{a_n}(\omega) = \tilde{H}_a(\omega)$ for all ω, proving right-continuity. As an inside, note also that

$$\tilde{H}_{a-} = \lim_{b\uparrow a} \tilde{H}_b = H_a = \inf\{t \geq 0\,;\, W(t) = a\}.$$

Next observe that

$$\tilde{H}_{b+a} = \tilde{H}_b + \inf\{t\,;\, W(\tilde{H}_b + t) - W(\tilde{H}_b) > a\}.$$

But we know from Lemma 2 and Theorem 1 that $\{W(\tilde{H}_b + t) - W(\tilde{H}_b)\,;\, t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}_{\tilde{H}_b+}$. It follows that $\{\tilde{H}_{b+a} - \tilde{H}_b\,;\, a \geq 0\}$ is independent of $\mathcal{F}_{H_b+}$ and has the same finite dimensional distribution as $\{\tilde{H}_a \;\; a \geq 0\}$. This proves the independence and stationarity of the increments of $\{\tilde{H}_a \;\; a \geq 0\}$, and finishes the proof that it is a Lévy process.

From Theorem 9, the density of H_a is $p(t) = a\dfrac{e^{-a^2/2t}}{\sqrt{2\pi}t^{3/2}}$, when $t > 0$. Therefore

$$\mathbb{E}\left[e^{iu\tilde{H}_a}\right] = a\int_0^\infty e^{iut}\frac{e^{-a^2/2t}}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t.$$

Because $\{\tilde{H}_a\,;\,a \geq 0\}$ is a Lévy process,

$$\mathbb{E}\left[e^{iu\tilde{H}_1}\right] = \left[\mathbb{E}\left[e^{iu\tilde{H}_{1/n}}\right]\right]^n$$

We will use this to compute the Lévy-Khinchine representation of $\mathbb{E}\left[e^{iu\tilde{H}_1}\right]$.

Since $\displaystyle\int_0^\infty a\frac{e^{-a^2/2t}}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t\,\mathrm{d}t = 1$,

$$\mathbb{E}\left[e^{iu\tilde{H}_a}\right] = 1 + a\int_0^\infty \left[e^{iut} - 1\right]\frac{e^{-a^2/2t}}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t.$$

This is very useful because $\left[e^{iut} - 1\right]\frac{1}{t^{3/2}}$ is integrable over $[0,\infty)$, since $e^{iut} - 1$ is of order t as $t \to 0$. Thus, as $a \to 0$, dominated convergence implies

$$\lim_{a\to 0}\int_0^\infty \left[e^{iut} - 1\right]\frac{e^{-a^2/2t}}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t = \int_0^\infty \left[e^{iut} - 1\right]\frac{1}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t.$$

For this reason,

$$\mathbb{E}\left[e^{iu\tilde{H}_1}\right] = \lim_{n\to\infty}\left[\mathbb{E}\left[e^{iu\tilde{H}_{1/n}}\right]\right]^n = \exp\{\int_0^\infty \left[e^{iut} - 1\right]\frac{1}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t\}.$$

This can be put into the standard format by writing it as

$$= \exp\{imu + \int_0^\infty \left[e^{iut} - 1 - iut\mathbf{1}_{[0,1)}(t)\right]\frac{1}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t\},$$

where

$$m = \int_0^1 \frac{t}{\sqrt{2\pi}t^{3/2}}\,\mathrm{d}t = \sqrt{\frac{2}{\pi}}.$$

Let $\triangle_t$ denote the Poisson point process on $[0,\infty) \times (0,\infty)$ associated to the measure

$$m(da)\frac{\mathrm{d}t}{\sqrt{2\pi}t^{3/2}},$$

where m denotes Lebesgue measure. Even though

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt{2\pi}t^{3/2}} = \infty, \tag{11}$$

since $\int_0^1 \frac{\mathrm{d}t}{\sqrt{2\pi}t^{3/2}} < \infty$, it may be shown that

$$Y(a) = \sum_{s \leq a} \triangle_s$$

converges for all t, almost-surely. This is left as an exercise. The process $\{Y(a)\,;\, a \geq 0\}$ is equal to $\{\tilde{H}_a\,;\, a \geq 0\}$ in distribution. Thus $\{\tilde{H}_a\}$ increases by jumps, distributed according to the characteristic measure $dt/(\sqrt{2\pi}t^{3/2})$. Because of 11, there are a countable number of jumps in any interval $s_1 < a < s_2$. If $\tilde{H}_a(\omega)$ jumps at a, $\tilde{H}_a(\omega) > \tilde{H}_{a-}(\omega) = H_a(\omega)$. Thus despite the fact that for each $a \geq 0$, $I\!\!P(H_a = \tilde{H}_a) = 1$ for each a, $\tilde{H}_a$, as a process in a increases only by jumps at which the two stopping times are not equal. This behavior reflects the roughness of Brownian path.

Finally, we note a bit of standard terminology. Non-decreasing Lévy processes are called *subordinators.*