Take-home exam

1. Go to http://www.math.rice.edu/∼dfield/dfpp.html, click on PPLANE 2002.2 and use this software to visualize the phase plane of a 2D system of ODE's. Choose "Linear System" in the "Gallery" menu. Set parameters $A = -2$, $B = -1$, $C = 1$ and $D = 2$.

   - Find the fixed point and perform the stability analysis (find eigenvectors and eigenvalues). Use the software to draw the phase plane. Click on the phase plane several times at different points to draw trajectories, especially in the vicinity of the fixed point. Sketch (or print out) the phase portrait. Indicate the trajectories corresponding to the eigenvectors.
   - Solve the initial value problem with $x(0) = -4$, $y(0) = 2$.
   - Click at point $(-4, 2)$ to draw the trajectory corresponding to your solution. Mark it on the sketch of the phase plane.

2. Using the same software as above, choose the "Competing species" in the "Gallery" menu. For the analysis below, assume that the coefficients A, B and C are positive.

   - What are the fixed points of the system (find expressions in terms of parameters A, B, C)? Perform stability analysis (find eigenvalues) of the system at the following points: (i) complete extinction ($x = 0, y = 0$), (ii) species "x" wins ($x > 0$, $y = 0$), (iii) species "y" wins ($y > 0$, $x = 0$).
   - Find the condition on the parameters of the system which guarantees that point (ii) is stable. Find the condition on the parameters of the system which guarantees that point (iii) is stable.
   - Set the parameters A, B and C such that point (ii) is stable and point (iii) is unstable. Set the "display Window" parameters such that all the fixed points are shown. Draw the phase plane. Click on the phase plane several times at different points to draw trajectories, especially in the vicinity of fixed points. Sketch (or print out) the phase portrait. What is the type of fixed points (i), (ii) and (ii)?

- Same as above, but choose the parameters such that point (iii) is stable and point (ii) is unstable.
- Same as above, but choose the parameters such that both point (iii) and point (ii) are stable.

3. Go to http://falstad.com/diffeq/. Choose LHS to be $ay'' + by' + cy$ and the RHS to be "Sine Wave". Set the parameters to be $a = 1$, $b = 0.1$, $c = 1$, $h = 3$ and $g = 0.25$. To change parameters, click and drag left or right (to increase or decrease). Do not worry if the parameters are not exactly as above, just make sure they are close to the given values. Set the initial point to be $y(0) = 0$ by finding a red cross and dragging it to the left end of the horizontal axis. Set "Initial y' " to 10.

    - Sketch (or print out) the solution together with the forcing (the sine wave).
    - Solve the initial value problem by Laplace transform. Show your work. You can either solve it in general (for general values of the parameters a, b, c, h, g and $y'(0)$, with $y(0) = 0$), or you can use the given values.
    - (extra credit) What is the limiting behavior of the solution as $t \to \infty$, assuming that $b \ll a, c$? Can you see it on the plot?

4. Go to http://falstad.com/fourier/. Press "Sawtooth" and then "Rectify". The resulting function has the period of 2π and is defined by

$$f(x) = \begin{cases} x, & 0 \le x \le \pi, \\ 0, & -\pi \le x < 0. \end{cases}$$

   Find Fourier Series for this function. Use the software to plot the Fourier Series with 2 terms and with 8 terms. Sketch (or print out) what you see. Use the plot with 2 terms to check your analytical result.

5. Using the same software as above, press "Noise".

    - Set the "number of terms" to 1. Look under "Magnitudes" and note the height of the bar (the size of the term). Then press "Rectify". What happens to the amplitude of the "zeroth" Fourier component? Why?

- "Noise" is a piecewise-constant function, with period 2π, whose values can be defined as follows. Split the interval $[-\pi, \pi]$ into N intervals of length $\Delta x = 2\pi/N$. The value $f(x)$ in each of the intervals is X_j, where X_j is a random variable taken from the uniform distribution between -1 and 1; in other words,

  $$f(x) = X_j \quad \text{for } -\pi + (j-1)\Delta x \le x < -\pi + j\Delta x, \quad 1 \le j \le N.$$

  Recall: the probability density function for the uniform distribution is given by

  $$g(x) = \begin{cases} 1/2, & -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

  What is the probability distribution for the "zeroth" Fourier component? What is its mean? What is the variance?

6. A Markov chain is given by the following transition matrix:

$$P = \begin{pmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/2 & 1/4 \end{pmatrix}$$

with entries $p_{ij} = p_{i \leftarrow j}$.

- What are transient and absorbing states of this system?
- What is the stationary probability distribution? Describe in words the final state of the system.
- What are the times of absorption starting from all transient states?
- What are the probabilities of absorption?