Homework 2

Dynamics of immunosuppressive viruses. Consider a model that contains two variables: the virus population, y, and the immune response that can control the infection in the long term, z. The model is given by the following pair of differential equations:

$$\dot{y} = ry\left(1 - \frac{y}{k}\right) - ay - pyz, \tag{1}$$

$$\dot{z} = \frac{cyz}{1 + \epsilon y} - qyz - bz. \tag{2}$$

The virus population grows at a density dependent rate $ry(1 - y/k)$. The parameter r can be considered to represent the rate of viral replication, while the parameter k represents the "carrying capacity". The virus population dies at a rate ay, and becomes inhibited by the immune response at a rate pyz. The immune response expands at a rate $cyz/(\epsilon y + 1)$. Thus, expansion is a saturating function of the amount of virus present. The virus population also inhibits the immune response at a rate qyz. Finally, in the absence of antigenic stimulation, the immune response declines at a rate bz.

The model is characterized by three outcomes. In the trivial case, S_0, there is no infection and no immune response ($y = z = 0$). Alternatively, the virus can establish an infection in the absence of an immune response, S_v, such that $y = k(1 - a/r)$, $z = 0$) (the "virus equilibrium"). Finally, the virus can establish an infection which is controlled by an immune response, S_i (the "immune control equilibrium"), with $y = y_1$ and $z = z_1$ with

$$y_{1,2} = \frac{c - q - b\epsilon \mp \sqrt{(c - q - b\epsilon)^2 - 4qb\epsilon}}{2\epsilon q},$$

$$z_{1,2} = \frac{1}{p}\left[r\left(1 - \frac{y_{1,2}}{k}\right) - a\right].$$

Note that equilibrium $\bar{S}_i$ given by $y = y_2$, $z = z_2$, is always unstable and therefore biologically irrelevant.

Show that depending on the value of r, the replication rate of the virus, different outcomes are possible. In particular, show that

(i) If $0 < r < a$, then the system converges to S_0.

(ii) If $a < r < a/(1 - y_1/k)$, the system converges to S_v.

(iii) If $a/(1 - y_1/k) < r < a/(1 - y_2/k)$, the system converges to S_i.

For each of these regimes, study the stability of the fixed points, S_0, S_v and S_i. For each fixed point:

- Linearize the system and find the eigenvalues of the linear matrix.
- Write down the general solution for the perturbation. Is this solution stable? Why?
- What kind of a fixed point is it? (Saddle? Node? Spiral?)
- If it is a saddle or a node, find the eigenvectors.

Sketch the phase portrait for each of the three regimes.