Math 16:642:621 – Homework 1* – Fall 2007 © Paul Feehan

This homework assignment explores a financial model with a finite market state space $\Omega := \{\omega_1, \dots, \omega_m\}$, a set of discrete trading times $\mathcal{T} := \{t_0, \cdots, t_n\}$ where $t_0 < \cdots < t_n$, with $\delta t := t_{k+1} - t_k$, $t_0 = 0$, $t_n = T$, and p assets with vector of pricing functions $A(t,\omega) := (S_1(t,\omega), \dots, S_p(t,\omega)) \in \mathbb{R}^p$, and initial prices $A(0) = (S_1(0), \dots, S_p(0))$ (independent of ω).

We often take one of the asset prices, say $S_1(t,\omega)$, to be the price $B(t)$ of a "riskless" asset (meaning independent of ω), the *bank* (or *money market*) *account* with constant interest rate $r \geq 0$, so that $B(t+\delta t) = B(t)(1 + r\delta t)$ (simple compounding) or $B(t+\delta t) = B(t)e^{r\delta t}$ (continuous compounding), for $t \in \mathcal{T}$; simple compounding is assumed in this assignment. When $p = 2$ (a two-asset model), we denote $S_2(t,\omega) = S(t,\omega)$ and think of S as the price of the stock or "risky" asset (value depends on ω). A *portfolio process* or *trading strategy* $\Phi(t,\omega) = (\phi_1(t,\omega), \dots, \phi_p(t,\omega)) \in \mathbb{R}^p$ is a vector of functions giving the number of shares of each asset, noting that $\Phi(t_1,\omega) = \Phi(0,\omega) = \Phi(0)$ (independent of ω). The *portfolio value process* is the dot product $\Pi(t,\omega) := \Phi(t,\omega) \cdot A(t,\omega) = \sum_{i=1}^p \phi_i(t,\omega) A_i(t,\omega)$.

For the rest of this assignment, suppose assume $n = 1$ (a *one-period model*), so that $\Phi = (\phi_1, \dots, \phi_p)$ (independent of (t,ω)). An *arbitrage* in this model is a portfolio Φ such that

(a) $\Pi(0) < 0$ and $\Pi(T,\omega_i) \geq 0$ for all $\omega_i \in \Omega$, or
(b) $\Pi(0) = 0$, $\Pi(T,\omega_i) \geq 0$ for all $\omega_i \in \Omega$, and $\Pi(T,\omega_j) > 0$ for some $\omega_j \in \Omega$,

where $\Pi(t,\omega) = \Phi \cdot A(t,\omega)$. The model is called *arbitrage-free* if no arbitrage portfolio exists.

For the next two problems, suppose $m = 2$ (a two-state model) and $p = 2$ and write $B(0) = B_0 > 0$, $S(0) = S_0 > 0$, $S(T,\omega_1) = dS_0$ (price of stock in down state), and $S(T,\omega_2) = uS_0$ (price of stock in up state), where $0 < d < u < \infty$; this is the *binomial branch model* (the *multiperiod* version is called the *binomial tree model*). We showed in class that if $1 + rT \leq d$, then the model admits an arbitrage.

1. Show that if $u \leq 1 + rT$, then the model admits an arbitrage. Conclude that if the binomial model is arbitrage-free, then $d < 1 + rT < u$.

2. Show that if $d < 1 + rT < u$, then the binomial model is arbitrage-free. Conclude that the binomial model is arbitrage-free if and only if $d < 1 + rT < u$.

3. Is it necessary to assume $d < 1 < u$ in order that the binomial branch model is arbitrage free? Explain why or why not.

Suppose $V(t,\omega)$ is an $\mathbb{R}$-valued function. We can think of $V(t,\omega)$ as the price function for a *derivative security*, a tradable asset which can be added to our set of p underlying assets, having *payoff* $V(T,\omega)$. Common examples include:

- $V(T,\omega) = S(T,\omega) - K$, a *forward contract*,
- $V(T,\omega) = (S(T,\omega) - K)^+$, a *European call option*,
- $V(T,\omega) = (K - S(T,\omega))^+$, a *European put option*,

where $x^+ = \max\{x, 0\}$, K is the contract strike, and S is a stock. In these examples, $V(T,\omega)$ is a known function of $S(T,\omega)$, and we wish to find $V(0) = V_0$.

We call Φ a *replicating portfolio* for V if

$$V(T,\omega) = \Phi \cdot A(T,\omega) \quad \text{for all } \omega \in \Omega.$$

If the model is arbitrage free, then one can prove that

$$V_0 = \Phi \cdot A(0). \tag{0.1}$$

Hence, if one has a replicating formula for the derivative security V and the model is arbitrage free, the initial price V_0 is uniquely determined. We ask you to show this directly below in the special case when $p = m = 2$.

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4. Consider the binomial model with underlying asset price functions $B(t)$, $S(t,\omega)$, and derivative security price function $V(t,\omega)$. Suppose $\Phi = (\phi_1, \phi_2)$ is a replicating portfolio for V, that is,

$$\begin{aligned} V(T,\omega) &= \phi_1 B(T) + \phi_2 S(T,\omega) \\ &= \phi_1 B_0(1+rT) + \phi_2 S(T,\omega), \quad \text{for all } \omega. \end{aligned}$$

Assume that both the binomial model with underlying assets $\{B, S\}$ and the augmented model with assets $\{B, S, V\}$ are arbitrage-free. Show that

$$V_0 = \phi_1 B_0 + \phi_2 S_0.$$

To see this, consider two cases: (1) $V_0 < \phi_1 B_0 + \phi_2 S_0$ and (2) $V_0 > \phi_1 B_0 + \phi_2 S_0$. For each case, show that an arbitrage exists by constructing a portfolio in $\{B, S, V\}$ with zero initial value, non-negative value at time T for all market states, and positive value for some market state.

5. Consider the binomial branch model with underlying asset price functions $B(t)$, $S(t,\omega)$, and derivative security price function $V(t,\omega)$. Suppose $\Phi = (\phi_1, \phi_2)$ is a replicating portfolio for V, that is,

$$V(T,\omega) = \phi_1 B_0(1+rT) + \phi_2 S(T,\omega), \quad \text{for all } \omega.$$

Assume that both the binomial model with underlying assets $\{B, S\}$ and the augmented model with assets $\{B, S, V\}$ are arbitrage-free, so by the preceding problem, one has

$$V_0 = \phi_1 B_0 + \phi_2 S_0.$$

Find the unknown (ϕ_1, ϕ_2) using the following steps:

(a) Show that (ϕ_1, ϕ_2) obeys the linear equation

$$\begin{pmatrix} B_0(1+rT) & S(T,\omega_1) \\ B_0(1+rT) & S(T,\omega_2) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} V(T,\omega_1) \\ V(T,\omega_2) \end{pmatrix}.$$

(b) Solve for (ϕ_1, ϕ_2):

$$\phi_2 = \frac{V(T,\omega_2) - V(T,\omega_1)}{S(T,\omega_2) - S(T,\omega_1)} \quad \text{and} \quad \phi_1 = \frac{1}{B_0(1+rT)} \left(V(T,\omega_1) - \phi_2 S(T,\omega_1) \right).$$

Denoting $S_d = S(T,\omega_1)$, $S_u = S(T,\omega_2)$, $V_d = V(T,\omega_1)$, and $V_u = V(T,\omega_2)$, the solution takes on a simpler form:

$$\phi_2 = \frac{V_u - V_d}{S_u - S_d} \quad \text{and} \quad \phi_1 = \frac{1}{B_0(1+rT)} \left(V_d - \phi_2 S_d \right).$$

(c) Show that

$$V_0 = \frac{1}{1+rT}(V_d - \phi_2 S_d) + \phi_2 S_0.$$

(d) Show that V_0 can also be written as

$$V_0 = \frac{1}{1+rT} \left[\left(\frac{1+rT-d}{u-d} \right) V_u + \left(\frac{u-(1+rT)}{u-d} \right) V_d \right]$$

(e) Assume that the binomial model is arbitrage free, so $d < 1 + rT < u$, and define

$$q_u := \frac{1+rT-d}{u-d} \quad \text{and} \quad q_d := \frac{u-(1+rT)}{u-d} = 1 - q_u.$$

Show that $0 < q_d < 1$ and

$$V_0 = D(T)[q_u V_u + q_d V_d],$$

where $D(t) := B(0)/B(t) = (1+rT)^{-1}$ is called the *discount factor*. [Note: In the case of continuous compounding, the arbitrage free condition becomes $d < e^{rT} < u$, and the same argument yields the preceding formula with $D(T) = e^{-rT}$.]

6. In the setup of the preceding problem, assume the derivative security is a forward contract with strike K, so $V(T,\omega) = S(T,\omega) - K$, where K is non-negative constant.

(a) Show that

$$\phi_2 = 1 \quad \text{and} \quad \phi_1 = \frac{-K}{B_0(1+rT)}.$$

(b) Conclude that

$$V_0 = -D(T)K + S_0.$$

[Note: We had assumed $B(T) = B_0(1+rT)$. If instead $B(t)$ grows at a continuously compounded rate r, then $B(T) = B_0 e^{rT}$ and the same argument would yield the preceding formula with $D(T) = e^{-rT}$. The *forward price* $F(0,T)$ of an asset S at time $t = 0$ for delivery at time T is defined to be that value of K for which $V_0 = 0$, that is, $F(0,T) = S_0/D(T) = e^{rT}S_0$ or $(1+rT)S_0$.]

7 (Optional)**.** Assume that the one-period, augmented market model with $p+1$ price functions $\{S_1, \ldots, S_p, V\}$ and m states is arbitrage free. Prove that Equation (0.1) must hold. [Hint: The argument is very similar to that for the special case of the 2-asset, 2-state binomial branch model.]