Math 16:642:621 – Homework 2[*] – Fall 2007 © Paul Feehan

This homework assignment continues to explore the one-period market model with finite state space $\Omega = \{\omega_1, \dots, \omega_m\}$, trading times $t \in \{0, T\}$, p assets with $p \times 1$ price vector $A(0) := (S_1(0), \dots, S_p(0))^*$ at time $t = 0$ (independent of ω), and at time T, the $p \times m$ price matrix $\mathcal{A}(T) := (S_i(T, \omega_j))$ or, equivalently, m price vectors $A(T, \omega_j) := (S_i(T, \omega_j))$, for $j = 1, \dots, m$.

If $\Phi = (\phi_1, \dots, \phi_p) \in \mathbb{R}^p$ is a portfolio with value $V(T, \omega) = \Phi A(T, \omega) = \sum_{i=1}^p \phi_i S_i(T, \omega)$ at time T, then its value at time $t = 0$ is $V(0) = \Phi A(0) = \sum_{i=1}^p \phi_i S_i(0)$. We call $\psi = (\psi_1, \dots, \psi_m)^*$ a *state price vector* if $\mathcal{A}(T)\psi = A(0)$ and $\psi_j > 0$ for all i. *First Fundamental Theorem of Asset Pricing*: "The one-period, finite state model is arbitrage-free if and only if it has a state price vector".

A *derivative security* (or *option* or *contingent claim*) is financial contract paying $X(\omega)$ at maturity in market state $\omega \in \Omega$, where $X : \Omega \to \mathbb{R}$ is called *payoff function*. A market is *complete* if and only if every derivative security is *attainable*, that is, it has a replicating portfolio. *Second Fundamental Theorem of Asset Pricing*: "If the one-period, finite state model is arbitrage-free, then it is complete if and only if its state price vector is unique".

Suppose a derivative security has payoff function $X(\omega)$. If a market is complete and arbitrage free, then X has a replicating portfolio, Φ, with value $V(T, \omega) := \Phi A(T, \omega) = X(\omega)$ at maturity. Because the market is arbitrage free, the price of the derivative security at time $t = 0$ must be $V(0)$ (otherwise there would be an arbitrage between the derivative security price and the portfolio of assets). Existence of a state price vector, ψ, implies that $V(0) = \Phi\mathcal{A}(T)\psi = V(T)\psi$, where $V(T) := (V(T, \omega_j), \dots, V(T, \omega_j))^*$, which yields the *pricing formula*:

$$V(0) = \sum_{j=1}^m \psi_j V(T, \omega_j).$$

The j-th *Arrow-Debreu security* has payoff function, e_j, defined by $e_j(\omega) = 1$ when $\omega = \omega_k$ and $e_k(\omega) = 0$ otherwise. If the market is complete and arbitrage free, then the j-th Arrow-Debreu security has price ψ_j at $t = 0$, for $j = 1, \dots, m$.

A *random variable* is a function, $X : \Omega \to \mathbb{R}$, on the state (or *sample*) *space* Ω. A *probability measure* $\mathbb{P}$ on a finite sample space Ω is a function which assigns to each subset $A \subset \Omega$ (or *event*) a number $\mathbb{P}(A) \in [0, 1]$ such that $\mathbb{P}(A) := \sum_{\omega \in A} \mathbb{P}(\omega)$ and $\mathbb{P}(\Omega) = 1$. The *expected* (or *mean* or *average*) *value* of X is $\mathbb{E}_{\mathbb{P}}[X] := \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$.

Let B be a *risk-free asset* in the market, with prices $B(0) = 1$ and $B(T) > 1$ independent of $\omega \in \Omega$, and let $D(t) := 1/B(t)$ denote the corresponding *discount factor*. A probability measure $\mathbb{Q}$ on Ω is called *risk-neutral* with respect to the discount factor, $D(T)$, if $S_i(0) = D(T)\mathbb{E}_{\mathbb{Q}}[S_i(T)]$ for $i = 1, \dots, p$. Equivalently, if $S_i^*(t, \omega) := D(t)S_i(t, \omega)$ are the *discounted asset price* processes (noting that $D(0) = 1$), then $\mathbb{Q}$ is risk-neutral if and only if $S_i^*(0) = \mathbb{E}_{\mathbb{Q}}[S_i^*(T)]$ for $i = 1, \dots, p$. A process $M(t, \omega)$ which obeys $M(0) = \mathbb{E}_{\mathbb{P}}[M(T)]$ is called a *martingale* with respect to $\mathbb{P}$. For this reason, a risk neutral measure $\mathbb{Q}$ is also called a *martingale measure*.

A state price vector ψ on Ω defines a probability measure $\mathbb{Q}$ on Ω by $\mathbb{Q}(\omega_j) := \psi_j/D(T)$, where $D(T) := \sum_{j=1}^m \psi_j$. The matrix equation $A(0) = \mathcal{A}(T)\psi$ is equivalent to the system $S_i(0) = \sum_{j=1}^m \psi_j S_i(T, \omega_j)$, for $i = 1, \dots, p$, and hence $S_i(0) = D(T)\sum_{j=1}^m \mathbb{Q}(\omega_j)S_i(T, \omega_j)$ and $\mathbb{Q}$ is a risk-neutral measure.

Conversely, given a discount factor, $D(T)$, and a risk-neutral measure $\mathbb{Q}$, one can define a vector ψ by $\psi_j := D(T)\mathbb{Q}(\omega_j) > 0$. The system of equations $S_i(0) = D(T)\mathbb{E}_{\mathbb{Q}}[S_i(T)]$ for $i = 1, \dots, p$ is equivalent to $A(0) = \mathcal{A}(T)\psi$ and thus ψ is a state price vector.

This leads to alternative formulations of the *First Fundamental Theorem of Asset Pricing*, "The one-period, finite state model is arbitrage-free if and only if it has a risk neutral measure", and the *Second Fundamental Theorem of Asset Pricing*: "If the one-period, finite state model is arbitrage-free, then it is complete if and only if the risk neutral measure is unique".

1. A *zero-coupon bond* is a derivative security with payoff $Z(T, \omega) = 1$.

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(a) Find a replicating portfolio Φ for Z.
(b) Find the $t = 0$ no-arbitrage price, $Z(0)$, in terms of ψ and $A(0)$.
(c) If R is the annually compounded risk free interest rate, write $Z(0)$ in terms of R.
(d) If r is the continuously compounded risk free interest rate, write $Z(0)$ in terms of r.

2. A market consists of IBM and MSFT stock, with closing stock prices \$80 and \$30 on September 11, 2007. Suppose that one year from now, there are only two market states, $\Omega = \{u, d\}$. If the market is up, the IBM, MSFT price changes are $+\$8, +\$6$; if the market is down, the price changes are $-\$2$, $-\$3$.

(a) Find a state price vector, $\psi = (\psi_1, \psi_2)$.
(b) Is the market arbitrage-free? Complete?
(c) If the market arbitrage-free, what is the annually compounded risk-free interest rate?

3. Consider a one-period market model with $m = 2$ states, $\{\omega_1, \omega_2\}$, and $p = 2$ assets with price functions $B(t)$ (money market account) and $S(t, \omega)$ (stock), trading times $t \in \{0, T\}$with $T = 1$, and prices $B(0) = 1$, $B(1) = 1 + r$ with $r = 1/9$, $S(0) = 5$, $S(1, \omega_1) = 20/3$, $S(1, \omega_2) = 40/9$.

(a) If possible, find a state price vector, $\psi = (\psi_1, \psi_2)^*$, and if not, explain.
(b) If possible, find a risk neutral measure, $\mathbb{Q}$, $\psi = (\psi_1, \psi_2)^*$, and if not, explain.
(c) Is the market arbitrage-free? Explain.
(d) If the market complete? Explain.

4. Consider a one-period market model with $m = 2$ states, $\{\omega_1, \omega_2\}$, and $p = 2$ assets with price functions $B(t)$ (money market account) and $S(t, \omega)$ (stock), trading times $t \in \{0, T\}$with $T = 1$, and prices $B(0) = 1$, $B(1) = 1 + r$ with $r = 1/9$, $S(0) = 5$, $S(1, \omega_1) = 20/3$, $S(1, \omega_2) = 40/9$.

(a) Suppose X is a derivative security defined by $X(\omega_1) = 7$, $X(\omega_1) = 2$. Is X attainable in the market? If yes, find a replicating portfolio, $\Phi = (\phi_1, \phi_2)$, and if not, explain.
(b) What is the no-arbitrage price of the security at time $t = 0$?

5. Consider a one-period market model with $m = 2$ states, $\{\omega_1, \omega_2\}$, and $p = 2$ assets with price functions $B(t)$ (money market account) and $S(t, \omega)$ (stock), trading times $t \in \{0, T\}$with $T = 1$, and prices $B(0) = 1$, $B(1) = 1 + r$ with $r = 1/9$, $S(0) = 5$, $S(1, \omega_1) = 20/3$, $S(1, \omega_2) = 40/9$. Suppose X is a call option on S with strike $K = 5$, maturity T, and payoff

$$X(\omega) = (S(T, \omega) - K)^+ = \max\{S(T, \omega) - K, 0\}.$$

(a) Compute the $t = 0$ price of of the option using the risk neutral pricing formula, $D(T)\mathbb{E}_{\mathbb{Q}}[X]$.
(b) Check your answer by finding a replicating portfolio, $\Phi = (\phi_1, \phi_2)$ with value $V(T, \omega) = \sum_{i=1}^{p} \phi_i S_i(T, \omega) = X(\omega)$ at $t = T$ and value $V(0) = \sum_{i=1}^{p} \phi_i S_i(0)$ at $t = 0$.

6. Consider a one-period market model with $m = 3$ states, $\{\omega_1, \omega_2, \omega_3\}$, and $p = 3$ assets with price functions $S_1(t, \omega) = B(t)$ (money market account) and $S_2(t, \omega)$, $S_3(t, \omega)$ (stocks), trading times $t \in \{0, T\}$with $T = 1$, and prices $B(0) = 1$, $B(1) = 1 + r$ with $r = 1/9$, $S_2(0) = 5$, $S_3(0) = 10$, and

$$\mathcal{A} = (S_i(T, \omega_j)) = \begin{pmatrix} 10/9 & 10/9 & 10/9 \\ 60/9 & 60/9 & 40/9 \\ 40/3 & 80/9 & 80/9 \end{pmatrix}.$$

(a) Find the system of equations obeyed by (q_1, q_2, q_3), if $q_j := \mathbb{Q}(\omega_j)$, for $i = 1, 2, 3$.
(b) Find the solution (q_1, q_2, q_3) and explain whether or not it is unique.
(c) If the solution $(q_1, q_2, q_3) = (1/2, 0, 1/2)$, explain whether the market is arbitrage free or not.
(d) Show that $\Phi := (\phi_1, \phi_2, \phi_3) = (0, x/2, -x/4)$ is an arbitrage for every $x > 0$.
(e) Is the market complete? Explain.