Math 16:642:621 – Homework 3* – Fall 2007 © Paul Feehan

Multiperiod market model. We consider a model with a finite market state space Ω, a finite set of trading times $\mathcal{T} := \{t_0, t_1, \dots, t_N\}$ with $1 \le N < \infty$, $0 = t_0 < t_1 < \cdots < t_N = T$; and a set of p assets with vector of price functions $A(t,\omega) := (S_1(t,\omega), \dots, S_p(t,\omega))^* \in \mathbb{R}^p$ and an initial price vector $A(t_0) := (S_1(t_0), \dots, S_p(t_0))^* \in \mathbb{R}^p$, independent of $\omega \in \Omega$.

A *stochastic process* is a function, $\mathcal{T} \times \Omega \to \mathbb{R}^d$, $(t,\omega) \mapsto X(t,\omega)$, for $d \ge 1$; for any fixed $\omega \in \Omega$, the function $\mathcal{T} \to \mathbb{R}$, $t \mapsto X(t,\omega)$ is a *sample path* of the process; for any fixed $t \in \mathcal{T}$, the function $\Omega \to \mathbb{R}$, $\omega \mapsto X(t,\omega)$ is a random variable.

A process $B(t,\omega)$ is a *numéraire* if $B(t,\omega) > 0$ for all (t,ω) and a *money market account* if the process is strictly increasing with t. The *interest rate* for the period $[t_{n-1}, t_n)$ is

$$r(t_n,\omega) := \frac{1}{t_n - t_{n-1}} \frac{B(t_n,\omega) - B(t_{n-1},\omega)}{B(t_{n-1},\omega)} > 0.$$

We shall assume that $B(t,\omega) = B(t)$ (independent of $\omega \in \Omega$) and $r(t) = r$ (a positive constant).

Tree models, adapted processes, and predictable processes. Assume $\Omega = \{(\xi_1, \dots, \xi_N) : \xi_i \in \{1, 2, \dots, b\}\}$, so each market state in Ω is represented by a *path* $\omega = (\xi_1, \dots, \xi_N)$, with b possibilities (or branches) for each ξ_n, so Ω has $|\Omega| = b^N$ elements. If $b = 2$, this is the sample space for the *binomial* tree model and if $b = 3$, this is the sample space for the *trinomial* tree model. If $b = 2$, we customarily denote the two choices for each ξ_n by any one of the pairs of labels $\{0, 1\}$, $\{-1, 1\}$, $\{d, u\}$, or $\{\text{down}, \text{up}\}$.

We denote $\omega|_{t_n} := (\xi_1, \dots, \xi_n)$, if $\omega := (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_N)$. A process $X(t,\omega)$ is *adapted* if $X(t_n,\omega) = X(t_n, \omega|_{t_n})$, for $n = 1, \dots, N$. A process $X(t,\omega)$ is *predictable* if $X(t_n,\omega) = X(t_n, \omega|_{t_{n-1}})$, for $n = 1, \dots, N$. A *trading strategy* (or *portfolio process*) is a predictable process $\Phi(t,\omega) := (\phi_1(t,\omega), \dots, \phi_p(t,\omega)) \in \mathbb{R}^p$, with $\Phi(t_1,\omega) = \Phi(t_0)$ (independent of ω); it has an associated *wealth* or *portfolio value process* $V(t_n,\omega) := \Phi(t_n,\omega) \cdot A(t_n,\omega)$ for $n > 0$ and $V(t_0) = \Phi(t_1) \cdot A(t_0)$ for $n = 0$. (The concepts of adapted and predictable processes may be defined, more generally, with respect to a choice of *filtration*, $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ of the set of all subsets of a market state space Ω, which we discuss later.)

Arbitrage and completeness. A trading strategy $\Phi(t,\omega)$ is *self-financing* if

$$V(t_n,\omega) = \Phi(t_{n+1},\omega) \cdot A(t_n,\omega), \quad n = 0, \dots, N, \quad \omega \in \Omega.$$

An *arbitrage* is a self-financing trading strategy for which either one of the following hold:

(a) $V(t_0) < 0$ and $V(T,\omega) \ge 0$ for all $\omega \in \Omega$, or
(b) $V(t_0) = 0$, $V(T,\omega) \ge 0$ for all $\omega \in \Omega$, and $V(T,\omega') > 0$ for some $\omega' \in \Omega$.

Some texts omit our requirement that the trading strategy be self-financing. The market is *complete* if every contingent claim $X(\omega)$ is *attainable* in the market, that is, it has a self-financing portfolio, $\Phi(t,\omega)$, with value process, $V(t,\omega)$, which *replicates* the claim: $X(\omega) = V(T,\omega)$ for all $\omega \in \Omega$.

The *binomial tree model process* is defined by constants $S_0 > 0$ and d, u, r with $0 < d < 1 + r\delta t < u$,

$$S(t_n,\omega) := u^{N(t,\omega)} d^{n - N(t,\omega)} S_0, \quad n = 0, \dots, N, \quad \omega \in \Omega,$$

where $N(t,\omega)$ is the number of ξ_k in $\omega|_{t_n}$ with $\xi_k = u$ and so $n - N(t,\omega)$ is the number of ξ_k in $\omega|_{t_n}$ with $\xi_k = d$

1. Suppose that the multi-period model trading times $t_0 < t_1 < \cdots < t_N$ is arbitrage-free. Show that each one-period model, with period $[t_{n-1}, t_n]$, must be arbitrage free, for $n = 0, \dots, N$.

2. Suppose $\Phi(t,\omega)$ is a trading strategy. Denote $\Delta A(t_n,\omega) := A(t_{n+1},\omega) - A(t_n,\omega)$ and $\Delta V(t_n,\omega) := V(t_{n+1},\omega) - V(t_n,\omega)$. Show that $\Phi(t,\omega)$ is self-financing if and only if

$$\Delta V(t_n,\omega) = \Phi(t_{n+1},\omega) \cdot \Delta A(t_n,\omega), \quad n = 0, \dots, N-1, \quad \omega \in \Omega.$$

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3. This exercise is a continuation of Problem 5 in Assignment 1. Show that the following identity holds *for any* $\omega \in \Omega$ and not just $\omega = d$, extending Problem 5(b) in Homework 1:

$$\phi_1 = \frac{1}{B_0(1+rT)} \left(V(T,\omega) - \phi_2 S(T,\omega)\right)$$

and hence that, for any $\omega \in \Omega$, the following holds, extending Problem 5(c) in Assignment 1,

$$V_0 = \frac{1}{1+rT}(V(T,\omega) - \phi_2 S(T,\omega)) + \phi_2 S_0,$$

where we use (ϕ_1, ϕ_2) to denote our replicating portfolio. It is customary to denote ϕ_2 by Δ.

4. Consider a three-period binomial tree model, with period length of one year, stock $S(t,\omega)$ with initial price $S_0 = 4$, $u = 2$, $d = 1/2$, and bank account with annually compounded interest rate $r = 1/4$. Let $X(\omega)$ denote the payoff of a European-style lookback option with maturity T equal to 3 years, and strike $K = 2$:

$$X(\omega) = \left(\max_{0\le t\le T} S(t,\omega) - K\right)^+.$$

Let $V(t,\omega)$ denote the value of a replicating portfolio.

(a) What are the stock prices $S(t,\omega)$ at each of the tree nodes?

(b) What are the no-arbitrage option values at each of the tree nodes? (Hint: Work backwards from the maturity date and repeatedly apply the results of the one-period model from Problem 5(d) in Assignment 1.)

(c) What are the values of the $\Delta(t_n,\omega)$ for $n = 0,1,2,3$ and all $\omega \in \Omega$?

(d) Verify that the following identity holds, for $n = 1,2,3$ and all $\omega \in \Omega$, by explicitly evaluating both sides:

$$V(t_n,\omega) = (1+r\delta t)\left[(V(t_{n-1},\omega) - \Delta(t_n,\omega)S(t_{n-1},\omega)\right] + \Delta(t_n,\omega)S(t_n,\omega).$$

Observe that at time t_{n-1}, you know $\Delta(t_n,\omega) = \Delta(t_n, \omega|_{t_{n-1}})$, even though you do not know whether $\xi_n = u$ or d.

Remark: The final identity above explains why Δ is called the *hedge ratio*. A trader's portfolio consisting of a long position in the option contract, with price V, and a short position of Δ shares of the stock, $-\Delta S$, will grow at the risk-free rate, r, from time t_{n-1} to t_n irrespective of whether the market moves up or down (that is, independent of whether $\xi_n = u$ or d):

$$V(t_n,\omega) - \Delta(t_n,\omega)S(t_n,\omega) = (1+r\delta t)\left[(V(t_{n-1},\omega) - \Delta(t_n,\omega)S(t_{n-1},\omega)\right].$$

5. This exercise asks you to use induction to extend the results of Problem 5 in Homework Assignment 1 from the one-period binomial model to the multiperiod binomial tree. (If you are unfamiliar with induction arguments, you may answer the questions asked when $N = 2$.) Our model consists of an asset $S(t,\omega) \in \mathbb{R}$, a set of trading times $t_0 < t_1 < \cdots < t_N = T$ with $\delta t = t_n - t_{n-1}$, a bank account $B(t_n) = (1+r\delta t)^n$, possible asset price moves $S(t_n,\omega) = uS(t_{n-1},\omega)$ or $dS(t_{n-1},\omega)$, where $0 < d < 1 + r\delta t < u$. (In your problem solution, you may assume $t_n = n$, $T = N$, and $\delta t = 1$ to simplify notation.)

(a) Use Problem 5(c) in Homework Assignment 1 to show that for $n = 1,\dots,N$ and all $\omega \in \Omega$,

$$V(t_{n-1},\omega) = \frac{1}{1+r\delta t}\left[(V(t_n,\omega) - \Delta(t_n,\omega)S(t_n,\omega)\right] + \Delta(t_n,\omega)S(t_{n-1},\omega),$$

where we now denote the predictable portfolio process $\phi_2(t_n,\omega)$ by

$$\Delta(t_n,\omega) := \frac{V(t_n;\omega|_{t_{n-1}},u) - V(t_n;\omega|_{t_{n-1}},d)}{S(t_n;\omega|_{t_{n-1}},u) - S(t_n;\omega|_{t_{n-1}},d)} = \frac{\partial V(t_n,\omega)}{\partial S(t_n,\omega)},$$

where we defined, for convenience, $\partial V(t_n,\omega) := V(t_n;\omega|_{t_{n-1}},u) - V(t_n;\omega|_{t_{n-1}},d)$ and $\partial S(t_n,\omega) := S(t_n;\omega|_{t_{n-1}},u) - S(t_n;\omega|_{t_{n-1}},d)$.

(b) Explain why your formula for $V(t_{n-1},\omega)$ depends only on $\omega|_{t_{n-1}}$ and not on whether $\xi_n = u$ or d, when $\omega = (\xi_1,\dots,\xi_N)$. Conclude that $V(t,\omega)$ is an adapted process.

(c) Show that for $n = 1, \dots, N$ and all $\omega \in \Omega$,

$$V(t_n,\omega) = \underbrace{(1+r\delta t)\overbrace{\left[V(t_{n-1},\omega) - \Delta(t_n,\omega)S(t_{n-1},\omega)\right]}^{\text{cash at } t_{n-1}}}_{\text{cash at } t_n} + \Delta(t_n,\omega)S(t_n,\omega).$$

(d) Use Problem 5(e) in Assignment 1 to show that, for $n = 1, \dots, N$ and all $\omega \in \Omega$,

$$V(t_{n-1},\omega) = \frac{1}{1+r\delta t}\left[(1-q)V(t_n;\omega|_{t_{n-1}},u) + qV(t_n;\omega|_{t_{n-1}},d)\right].$$

(Hint: You will find it easiest to start with $n = N$ and work backward using induction.)

(e) Define a probability measure on the sample space $\Omega = \{\omega = (\xi_1, \dots, \xi_N) : \xi = u \text{ or } d\}$ by $\mathbb{Q}(\omega) := (1-q)^k q^{N-k}$, where k is the number of ξ_n in ω with $\xi_n = u$, and $0 < q < 1$ is

$$q := \frac{u - (1+r\delta t)}{u-d}.$$

Use induction to show

$$V(0) = \frac{1}{(1+r\delta t)^N}\sum_{\omega\in\Omega}\mathbb{Q}(\omega)V(T,\omega),$$

where $D(T) := 1/B(T)$ and $B(T) = (1+r\delta t)^N$, and hence that

$$V(0) = D(T)\mathbb{E}_{\mathbb{Q}}\left[V(T)\right].$$

This is the *risk-neutral pricing formula* for the multiperiod model.

6 (Optional)**.** Consider the one-period model with finite state space Ω and finitely many assets with p price functions $S_j(t,\omega) \in \mathbb{R}$, $j = 1, \dots, p$ (not necessarily positive and none assumed to be "riskless", that is, independent of ω), where $t = t_0$ or t_1.

(a) Suppose that $S_1(t,\omega) = B(t)$, where $B(t_0) = B_0 > 0$ and $B(t_1) = (1+r\delta t)B_0$, where $r > 0$ and $\delta t = t_1 - t_0$. Show that if Φ is a Type A arbitrage, then there must a Type B arbitrage $\tilde{\Phi}$. Conclude that when one asset is a bank account, a one-period model is arbitrage-free if and only if there is no Type B arbitrage.

(b) Suppose, more generally, that $S_1(t,\omega) = N(t,\omega)$ is a numéraire. Show that if Φ is a Type A arbitrage, then there must a Type B arbitrage $\tilde{\Phi}$. Conclude that when one asset is a numéraire, a one-period model is arbitrage-free if and only if there is no Type B arbitrage.

7 (Optional)**.** Consider the one-period model with finite state space Ω and finitely many assets. Suppose that $S_j(t,\omega) \in \mathbb{R}$, $j = 1, 2$ are the prices of two assets. We say *linear pricing* holds if for all $\phi_1, \phi_2 \in \mathbb{R}$, we have that $\phi_1 S_1(t_0) + \phi_2 S_2(t_0)$ is the value of the security at time t_0 with payoff $\phi_1 S_1(t_1,\omega) + \phi_2 S_2(t_1,\omega)$ at time t_1. Show that linear pricing holds if and only if the market has no Type A arbitrage.

8 (Optional)**.** Suppose a multi-period model with finite market state space Ω and a set of p assets with price functions $S_j(t,\omega)$, $j = 1, \dots, p$, is arbitrage free. Let $C(\omega)$ be a "contingent claim", that is, a function of $\omega \in \Omega$ whose value is revealed at time $t = T$ (but not necessarily at time $t < T$, as $\omega|_t$ may be insufficient to determine $C(\omega)$). Suppose $\Phi(t,\omega)$ is a replicating portfolio for $C(\omega)$, so that $C(\omega) = V(T,\omega)$, if $V(t,\omega) := \Phi(t,\omega)\cdot A(t,\omega)$, for all (t,ω). Suppose $C(t,\omega)$ is a price function for the security with payoff $C(\omega)$. Explain why the market with $p+1$ assets with price functions $S_1(t,\omega), \dots, S_p(t,\omega), V(t,\omega)$ is arbitrage free if and only if $C(t,\omega) = V(t,\omega)$ for all (t,ω). (Remark: This is the *Principle of No-Arbitrage Pricing*.)