Math 16:642:621 – Homework 4* – Fall 2007 © Paul Feehan

Probability measures and random variables. Suppose Ω is a non-empty set; this will serve as the *sample* or *market state space* in our financial models. A set $\mathcal{F}$ of subsets of Ω is called a sigma algebra if it (i) contains the empty set, $\{\,\} \in \mathcal{F}$, (ii) is closed under complements, $\Omega \setminus A \in \mathcal{F}$ whenever $A \in \mathcal{F}$, and (iii) closed under countable unions, $\cup_{n=1}^{\infty} A_i \in \mathcal{F}$ whenever $A_i \in \mathcal{F}$, for all integers $i \geq 1$. A sigma algebra for Ω is contained in but may be strictly smaller than the *power set* of Ω, the set of all subsets of Ω. The pair $(\Omega, \mathcal{F})$ is called a *measurable space.* If $\Omega = \mathbb{R}$, the sigma algebra generated by all closed intervals in $\mathbb{R}$ is called the *Borel sigma algebra* for $\mathbb{R}$ and denoted by $\mathcal{B}(\mathbb{R})$. A *random variable* on $(\Omega, \mathcal{F})$ is a function, $X : \Omega \to \mathbb{R}$, which is $\mathcal{F}$-measurable, that is, $X^{-1}(B) \equiv \{X \in B\} \in \mathcal{F}$ whenever $B \in \mathcal{B}(\mathbb{R})$ and $X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$. (One can show that function X is $\mathcal{F}$-measurable if and only if $\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$, a criterion which is more easily verified.) A *probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0,1]$ such that(i) $\mathbb{P}(\Omega) = 1$ and (ii) $\mathbb{P}(\cup_{n=1}^{\infty} A_i) = \sum_{n=1}^{\infty} \mathbb{P}(A_i)$ if $A_i \in \mathcal{F}$ for all $i \geq 1$ and $A_i \cap A_j = \{\,\}$ for all $i \neq j$. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.* The *probability distribution measure* on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a function $\mu_X : \mathcal{B}(\mathbb{R}) \to [0,1]$ induced by a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and given by $\mu_X(B) = \mathbb{P}(X^{-1}(B))$ for all $B \in \mathcal{B}(\mathbb{R})$; the *probability distribution function* $F_X : \mathbb{R} \to [0,1]$ is given by $F_X(x) := \mathbb{P}\{X \leq x\}$, for all $x \in \mathbb{R}$; one calls $f_X : \mathbb{R} \to \mathbb{R}$ a *probability density function* if and $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$; one can show that $f_X \geq 0$ (with probability one) and $\int_{-\infty}^{\infty} f_X(y)\,dy = 1$.

Expectations. Suppose $X : \Omega \to \mathbb{R}$ is a random variable on a probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. If Ω is countable, then the *expected value* of X is $\mathbb{E}[X] := \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$; otherwise, $\mathbb{E}[X] := \int_\Omega X\, d\mathbb{P}$, the *integral* of X with respect to $\mathbb{P}$. If X is a bounded function on $\Omega = [a,b]$ with $-\infty < a < b < \infty$ and the *Riemann integral,* $\int_a^b X(x) f_X(x)\,dx$, is well-defined, then one can show that $\mathbb{E}[X] = \int_a^b X(x) f_X(x)\,dx$. For more general sample spaces, Ω, or random variables, X, or probability measures, $\mathbb{P}$, the expected value, $\int_\Omega X\, d\mathbb{P}$, is defined by the *Lebesgue integral.*

1. Consider the binomial tree model of Assignment 3, with stock process $S(t,\omega)$, money market account $B(t)$, and trading times $t_0 < t_1 < \cdots < t_N = T$ with $\delta t = t_{n+1} - t_n$. Suppose that interest is compounded continuously over $[t_0, T]$ with rate $r \geq 0$ so that $B(t_{n+1}) = e^{r\delta t} B(t_n)$ (rather than $(1 + r\delta t)B(t_n)$, for simple compounding at each t_n), while $S(t_{n+1}, \omega) = uS(t_n, \omega)$ when $\xi_{n+1} = u$ and $dS(t_n, \omega)$ when $\xi_{n+1} = d$, where $\omega = (\xi_1, \ldots, \xi_N)$. Recall that $q = (u - e^{r\delta t})/(u - d)$. An important special case of the binomial model is defined by the choices

$$u := e^{(r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}} \quad \text{and} \quad d := e^{(r - \frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}}.$$

The parameter $\sigma > 0$ is called the *volatility* of the stock.

(a) Show that

$$q = \frac{e^{-\frac{1}{2}\sigma^2\delta t + \sigma\sqrt{\delta t}} - 1}{e^{-\frac{1}{2}\sigma^2\delta t + \sigma\sqrt{\delta t}} - e^{-\frac{1}{2}\sigma^2\delta t - \sigma\sqrt{\delta t}}} = \frac{e^{\sigma\sqrt{\delta t}} - e^{\frac{1}{2}\sigma^2\delta t}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}},$$

$$1 - q = \frac{e^{\frac{1}{2}\sigma^2\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}},$$

and hence that

$$\lim_{\delta t \to 0} q(\delta t) = \frac{1}{2},$$

for any $\sigma > 0$. [Hint: Recall L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

whenever the limit on the left exists and the functions $f(x)$, $g(x)$ are differentiable at $x = 0$.]

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(b) Show that the binomial tree model is arbitrage free (that is, $0 < d < e^{r\delta t} < u$) if and only if $0 < q < 1$ and show that $0 < q < 1$ if and only if $\sigma < 2/\sqrt{\delta t}$.

(c) Suppose $r = 5\%$ per year and that Google (Nasdaq: GOOG) has volatility $\sigma = 20\%$ per year and closing stock price today of $S_0 = \$570$. Using a 2-period binomial model with period length one year, what are the possible stock prices at each node of the binomial tree?

(d) What is the price today of a European-style call option written on the underlying stock, Google, with maturity $T = 2$ years and strike $K = \$570$?

(e) Suppose you sell the preceding call option contract for Google to a client (you are then *short* the call option). How many shares of Google should you hold during each time period to ensure that your resulting portfolio is riskless?

(f) Suppose you instead buy the preceding call option contract for Google from a market maker (you are then *long* the call option). How many shares of Google should you hold during each time period to ensure that your resulting portfolio is riskless?

2. Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space, let $X : \Omega \to \mathbb{R}$ be an integrable random variable, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a *convex* function (that is, $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$ for all $0 \leq t \leq 1$ and $x, y \in \mathbb{R}$). Assume φ is continuous. Then *Jensen's Inequality* states that $\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$. [Jensen's Inequality is still true even when φ is just assumed to be measurable.] Consider the arbitrage-free, one-period binomial model of the preceding problem, with *discount factor* $D(T) = e^{-rT}$ (continuous compounding) or $D(T) = (1 + rT)^{-1}$ (discrete compounding at time T).

(a) Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = (x - K)^+$, for some constant K. Show that φ is convex.

(b) Let $C(0, T, S, K)$ be the price at $t = 0$ of a European-style call option for the stock S with strike K and maturity T, so it has payoff $(S(T) - K)^+$. Recall that

$$C(0, T, S, K) = D(T)\mathbb{E}\left[(S(T) - K)^+\right].$$

Explain why

$$C(0, T, S, K) \geq (S_0 - D(T)K)^+ .$$

(c) Let $F(0, T, S, K)$ be the price at $t = 0$ of a forward contract for the stock S with strike K and maturity T, so it has payoff $S(T) - K$. Explain why

$$F(0, T, S, K) = D(T)\mathbb{E}[S(T) - K] = S_0 - D(T)K.$$

3. A European-style call with maturity T and strike K written on a stock $S(t)$ has payoff $(S(T)-K)^+$ and a European-style put has payoff $(K - S(T))^+$.

(a) Show that $(x - K)^+ - (K - x)^+ = x - K$, for any $x \in \mathbb{R}$.

(b) Show that

$$(S(T) - K)^+ - (K - S(T))^+ = S(T) - K,$$

where the right-hand side is the payoff of the forward contract.

(c) Let $C(t, T, S, K)$ be the price at time $0 \leq t \leq T$ of a European-style call option with strike K and maturity T written on the stock S and let $P(t, T, S, K)$ be the price of the put option. Show that

$$C(0, T, S, K) - P(0, T, S, K) = S_0 - R^{-1}K,$$

where $R = e^{rT}$ or $1 + rT$. [This relationship is called *put-call parity*; the formula does not depend on the stochastic process model used to describe the stock price, though you may assume the binomial model if you wish, in order to solve the problem.]

4. Suppose that you are given a one-period model with maturity $T > 0$ and stock price

$$S(T) = S_0 e^{Z(T)},$$

where $Z(T)$ is a normal random variable with mean $\mathbb{E}[Z(T)] = \mu T$ and variance $\mathbb{E}[(Z(T) - \mu T)^2] = \sigma^2 T$, where μ and $\sigma > 0$ are constants. Recall that the probability density function $\phi(x)$ for a normal random variable X with mean μ and variance σ^2 is

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$$

and that, for any integrable function f,

$$\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx.$$

(a) Use calculus to verify that

$$\int_{-\infty}^{\infty} \phi(x)\,dx = 1.$$

(b) Show that $\mathbb{E}[S(T)] = S_0 e^{\mu T + \frac{1}{2}\sigma^2 T}$.

(c) What is $\mathbb{E}[S(T)]$ when $\mu = r - \frac{1}{2}\sigma^2$?

5 (Optional)**.** Suppose that a multi-period model with finite state space Ω, assets $S_1(t,\omega), \ldots, S_p(t,\omega)$, and trading times $t_0 < t_1 < \cdots < t_N$ is arbitrage-free. Suppose $S_1(t,\omega) = B(t)$ is a mony market account, with risk-free rate r. Show that each one-period model, for the periods $[t_{i-1}, t_n]$ must be arbitrage free, for $i = 0, \ldots, n$. [The assumption that one asset was a money market account was omitted in Assignment 3, so we are providing you with another opportunity to solve the problem. Assume an arbitrage exists for the period $[t_{i-1}, t_n]$ and obtain an arbitrage for the period $[t_0, T]$ by investing risklessly before t_{i-1} and after t_n, yielding a contradiction.]