Math 16:642:621 – Homework 5[*] – Fall 2007 © Paul Feehan

Notes on Conditional Expectation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a random variable. The σ-algebra $\sigma(X)$ generated by a random variable X is the collection of all subsets of the form $\{\omega \in \Omega : X(\omega) \in B\}$, where B ranges over all (Borel-measurable) subsets of $\mathbb{R}$. If $\mathcal{G} \subset \mathcal{F}$ is another σ-algebra, we say that X is *$\mathcal{G}$-measurable* if $\sigma(X) \subset \mathcal{G}$. The *conditional expectation of X given $\mathcal{G}$* is a random variable characterized by the following two properties:

(a) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable,
(b) $\mathbb{E}[X|\mathcal{G}]$ has the *partial averaging property*:

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega)\, d\mathbb{P}(\omega) = \int_A X(\omega)\, d\mathbb{P}(\omega), \quad \text{for all } A \in \mathcal{G}.$$

If $\mathcal{G} = \sigma(Y)$ for another random variable Y, we denote $\mathbb{E}[X|Y] = \mathbb{E}[X|\mathcal{G}]$.

Two events $A, B \in \mathcal{F}$ are *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Two σ-algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathcal{G}$, $B \in \mathcal{H}$. Two random variables X, Y are *independent* if their σ algebras $\sigma(X)$, $\sigma(Y)$ are independent.

We describe some examples to make the concept of conditional expectation more accessible.

Example 1. Suppose X, Y are random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite Ω. Then

$$\mathbb{E}[X|Y = y] := \sum_x x\mathbb{P}\{X = x|Y = y\} = \sum_x x\frac{\mathbb{P}\{\{X = x\} \cap \{Y = y\}\}}{\mathbb{P}\{Y = y\}}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be the function $g(y) := \mathbb{E}[X|Y = y]$ and define

$$\mathbb{E}[X|Y] := g(Y).$$

One can show that this construction of $\mathbb{E}[X|\mathcal{G}]$ is the conditional expectation in the sense of Definition 2.3.1 in Shreve II when $\mathcal{G} = \sigma(Y)$.

Example 2. Let (X, Y) be a pair of random variables with joint density function $f_{X,Y}(x, y)$ and marginal density $f_Y(y)$. The *conditional density* for X given $Y = y$ is

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x, y)}{f_Y(y)},$$

and

$$\mathbb{E}[X|Y = y] := \int_{\mathbb{R}} y f_{X|Y}(x|y)\, dx.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be the function $g(y) := \mathbb{E}[X|Y = y]$ and define

$$\mathbb{E}[X|Y] := g(Y).$$

Since Y is a random variable, so is $g(Y)$ and it is $\sigma(Y)$-measurable. In Exercise II.2.10 in Shreve, you are asked to show that $\mathbb{E}[X|Y]$ has the partial averaging property and so is a conditional expectation in the sense of Definition 2.3.1 in Shreve II when $\mathcal{G} = \sigma(Y)$.

Example 3. Suppose X is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite Ω. Following Pliska (1997), given $A \in \mathcal{F}$, define

$$\mathbb{E}[X|A] := \sum_x x\mathbb{P}\{X = x|A\} = \sum_x x\frac{\mathbb{P}\{\{X = x\} \cap A\}}{\mathbb{P}\{A\}}.$$

Bayes' Theorem implies that

$$\mathbb{E}[X|A] = \sum_{\omega \in A} X(\omega)\frac{\mathbb{P}\{\omega\}}{\mathbb{P}\{A\}} = \sum_{\omega \in \Omega} 1_A(\omega)X(\omega)\frac{\mathbb{P}\{\omega\}}{\mathbb{P}\{A\}} = \frac{\mathbb{E}[1_A X]}{\mathbb{P}\{A\}}.$$

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[*] Last update: October 20, 2007

Suppose $\mathcal{G} \subset \mathcal{F}$ is a sigma algebra and let $\mathcal{P}$ be the partition of Ω into disjoint subsets corresponding to the sigma algebra $\mathcal{F}$. Then

$$\mathbb{E}[X|\mathcal{G}](\omega) = \sum_{A \in \mathcal{P}} \mathbb{E}[X|A] 1_A(\omega), \quad \omega \in \Omega.$$

One can show that this construction of $\mathbb{E}[X|\mathcal{G}]$ is the conditional expectation in the sense of Definition 2.3.1 in Shreve II.

Example 4. Consider the N-period binomial model with stock price process $S_t(\omega)$, bank account B_t with annually compunded interest rate r, trading times $t = 0, 1, \dots, N$ years, and $0 < d < 1+r < u$. The probability space is $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\Omega = \{(\xi_1, \dots, \xi_N) : \xi_i = u \text{ or } d\}$ and $\mathcal{F}$ is the set of all subsets of Ω. The σ-algebra $\mathcal{F}_n$ gives the information obtained by all possible stock price moves up to time $t = n$, so $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_1 = \{\varnothing, A_u, A_d, \Omega\}$ with $A_u = \{u\}$, $A_d = \{d\}$, and so on, and form a *filtration* $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_N = \mathcal{F}$. We can consider $\mathcal{F}_n \subset \mathcal{F}$ by identifying $(\xi_1, \dots, \xi_n) \in \mathcal{F}_n$ with $\{(\xi_1, \dots, \xi_n, \eta_{n+1}, \dots, \eta_N) : \eta_j = u \text{ or } d, \text{where } n+1 \le j \le N\} \in \mathcal{F}$. The probability measure $\mathbb{Q} : \mathcal{F}_n \to [0, 1]$ is defined using

$$\mathbb{Q}(\xi_1, \dots, \xi_n) = (1-q)^{\#\{\xi_j = u\}} q^{\#\{\xi_j = d\}},$$

where $\#\{\xi_j = u\}$ is the number of $\xi_j = u$, for $1 \le j \le n$, and $\#\{\xi_j = d\}$ is similarly defined, and $0 < q < 1$ is the risk-neutral probability defined in Problem 4(e) in Assignment 1: $q = (u-1-r)/(u-d)$. Let $X : \Omega \to \mathbb{R}$ be a random variable. The *conditional expectation of X given $\mathcal{F}_n$* is the random variable

$$\mathbb{E}[X|\mathcal{F}_n](\xi_1, \dots, \xi_n) := \sum_{(\xi_{n+1}, \dots, \xi_N)} \mathbb{Q}(\xi_{n+1}, \dots, \xi_N) X(\xi_1, \dots, \xi_N),$$

or, more concisely,

$$\mathbb{E}[X|\mathcal{F}_n](\omega|_n) = \sum_{\bar{\omega}|_n} \mathbb{Q}(\bar{\omega}|_n) X(\omega),$$

where $\omega|_n = (\xi_1, \dots, \xi_n)$ and $\bar{\omega}|_n = (\xi_{n+1}, \dots, \xi_N)$, if $\omega = (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_N)$. Note that $|\Omega|_n| = 2^n$ and $|\mathcal{F}_n| = 2^{2^n}$

1. A trader at the American Options Exchange buys and sells European-style call options on the S&P500 index. Today the value of one "share" of the S&P500 is 1,538 and the only two possible values for the S&P500 one year from today are 1,692 and 1,384. The trader uses historical data to estimate that, one year from now, the probabilities of the S&P500 having values 1,692 and 1,384 are $1 - p = 0.9$ (up move) and $p = 0.1$ (down move), respectively. He computes the price, $V_{\mathbb{P}}(0)$, of a call option with payoff $V(T) := (S(T) - K)^+$ using this "historical" measure $\mathbb{P}$ with $\mathbb{P}(\{d\}) = p$ and $\mathbb{P}(\{u\}) = 1 - p$:

$$V_{\mathbb{P}}(0) = D(T)\mathbb{E}_{\mathbb{P}}[V(T)],$$

where $S(t, \omega)$ is the value of the S&P500, $D(T) = (1 + rT)^{-1})$ is the discount factor and $T = 1$. You instead use the risk-neutral probability measure $\mathbb{Q}$ to compute the no-arbitrage price of the call option and which is the price quoted by the Chicago Board Options Exchange:

$$V_{\mathbb{Q}}(0) = D(T)\mathbb{E}_{\mathbb{Q}}[V(T)],$$

where $\mathbb{Q}(\{d\}) = q$ and $\mathbb{Q}(\{u\}) = 1 - q$, with $q = (u - 1 - r)/(u - d)$ and annually compounded risk-free interest rate $r = 5\%$. Assume a strike of $K = 1,615$.

(a) Compute $\mathbb{E}_{\mathbb{P}}[S(T)]$ and $V_{\mathbb{P}}(0)$.
(b) Compute q, $\mathbb{E}_{\mathbb{Q}}[S(T)]$ and $V_{\mathbb{Q}}(0)$.
(c) Describe an arbitrage portfolio involving call option positions with a potential payoff of \$1,000. [Hints: You may consider call options quoted with price $V_{\mathbb{P}}(t)$ and price $V_{\mathbb{Q}}(t)$ by different exchanges as different assets having the same payoff $V(T) = (S(T) - K)^+$.]

2. Consider the N-period binomial model with stock price process $S_t(\omega)$ defined by $N = 3$, $S_0 = 4$, $u = 2$, $d = 1/2$, and $r = 1/4$ (annually compounded), with δt equal to one year and $\Omega = \{(\xi_1, \xi_2, \xi_3) : \xi_j = u \text{ or } d\}$.

(a) Find all values of $S_t(\omega)$, $t = 0, 1, 2, 3$, and draw the tree of stock price values.
(b) List the elements of each of the σ-algebras $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$.
(c) List the elements of each of the σ-algebras $\sigma(S_0)$, $\sigma(S_1)$, and $\sigma(S_2)$. [Hint: To find $\sigma(S_2)$, first compute all subsets of Ω of the form $\{\omega \in \Omega : S_2(\omega) = s\}$, where $s = 1$, 4, or 16 and then include all additional sets obtained by taking complements or unions.]
(d) Is $\sigma(S_2) = \mathcal{F}_2$?
(e) Are S_2 and S_1 independent? Explain why or why not heuristically using the stock price tree and confirm your answer with a justification or counterexample.
(f) Are S_2/S_1 and S_1 independent? Explain why or why not heuristically using the stock price tree and confirm your answer with a justification or counterexample.
(g) Compute $\mathbb{E}[S_3|\mathcal{F}_n](\omega)$, for $n = 0, 1, 2, 3$.
(h) Compute $\mathbb{E}[S_3|\sigma(S_n)](\omega)$, for $n = 0, 1, 2, 3$.
(i) Compare $\mathbb{E}[S_3|\mathcal{F}_2](\omega)$ and $\mathbb{E}[S_3|\sigma(S_2)](\omega)$: Are they equal for all $\omega \in \Omega$?

3. For the N-period binomial model, we say that an adapted process $M_n(\omega)$, $0 \le n \le N$, $\omega \in \Omega$, is a *martingale* if $\mathbb{E}_{\mathbb{P}}[M_n|\mathcal{F}_m] = M_m$ for all $0 \le m \le n \le N$, with respect to the filtration $\{\mathcal{F}_n\}_{n\ge 0}$ and a measure $\mathbb{P}$ on $(\Omega, \mathcal{F}) := (\Omega_\infty, \mathcal{F}_\infty)$, the measurable space of Example 1.1.4 in Shreve II, with $\mathcal{F}_\infty$ the sigma-algebra generated by $\cup_{n=0}^{\infty}\mathcal{F}_n$.

Define a process $M_n(\omega) := (1 + r)^{-n}S_n(\omega)$, $0 \le n \le N$. Show that $M_n(\omega)$ is a martingale with respect to the filtration $\mathcal{F}_n$ and risk-neutral measure $\mathbb{Q}$, when $N = 3$ and $S_0 = 4$, $u = 2$, $d = 1/2$, and $r = 1/4$ (so that $\mathbb{Q}\{u\} = \mathbb{Q}(d) = 1/2$). [Hints: You may assume that it is enough to check the cases $n = m + 1$, $m = 0, 1, 2$.]

4. For the N-period binomial model with $\delta t = 1$, we say that an adapted process $X_n(\omega)$, $0 \le n \le N$, $\omega \in \Omega$, is *Markov* if for every function $f(x)$ then $\mathbb{E}[f(X_n)|\mathcal{F}_m] = \mathbb{E}[f(X_n)|X_m]$ for all $0 \le m \le n \le N$. Show that $S_n(\omega)$ is Markov for the special case $f(x) = x$, when $N = 3$, $S_0 = 4$, $u = 2$, and $d = 1/2$. [Hints: You may assume that it is enough to check the cases $n = m + 1$, $m = 0, 1, 2$.]

5 (Optional)**.** Let X, Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with joint density $f_{X,Y}(x, y)$, let $g(x) := \mathbb{E}[Y|X = x]$, and define $\mathbb{E}[Y|X] := g(X)$. Show that the partial averaging property holds:

$$\int_A g(X)\, d\mathbb{P} = \int_A Y\, d\mathbb{P}, \quad \text{for all } A \in \sigma(X).$$

[Hints: Use $\mathbb{E}[h(X)] = \int_{\mathbb{R}} h(x)f_X(x)\, dx$ and $\mathbb{E}[h(X, Y)] = \int_{\mathbb{R}} h(x, y)f_{X,Y}(x, y)\, dxdy$, and fact that $A = X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$ for some Borel subset $B \subset \mathbb{R}$, by definition of $\sigma(X)$. See Exercise II.2.10 and Definition 2.3.1 (ii) in Shreve II; note that the roles of X and $\mathcal{G} = \sigma(Y)$ are interchanged in the definition and the exercise.]