

1. A European-style *digital* call option on an underlying  $S(t)$ ,  $t \in [0, T]$ , with strike  $K$  has payoff

$$DC(T, S, K) = \begin{cases} 1 & \text{if } S(T) > K, \\ 0 & \text{if } S(T) \leq K. \end{cases}$$

A European-style *digital* put is similarly defined, except that now  $DP(T, S, K) = 1$  if  $K > S(T)$  and is zero otherwise.

- Sketch the graphs of the payoffs of a standard European call option,  $C(T, S, K)$ , and the digital call as functions of the terminal stock price,  $S(T)$ .
- Consider the binomial model with  $T = 3$ ,  $\delta t = 1$ ,  $S_0 = 4$ ,  $u = 2$ ,  $d = 1/2$ , annually compounded risk-free interest rate  $r = 1/4$ , and stock process  $S(t) = S(t, \omega)$ . Compute the tree of stock prices  $S(t)$ . If  $K = 6$ , compute the corresponding no-arbitrage digital call option values,  $V(t) = V(t, \omega)$ , and in particular,  $V(0)$ .
- Compute the tree of hedge ratio process values,  $\Delta(t) = \Delta(t, \omega)$ .
- Explain how to “delta” hedge the option,  $V(t_i)$ , by ensuring that at time  $t_{i-1}$  you hold a suitable number of shares of the stock  $S(t_i)$ . [Hint: See Problem 4(d) in Assignment 3.]

2. A European-style (*bull*) *spread* call option on a stock  $S(t)$ , with strikes  $K_1 < K_2$ , has payoff

$$\begin{aligned} H(T, S, K_1, K_2) &= C(T, S, K_1) - C(T, S, K_2) \\ &= \begin{cases} K_2 - K_1 & \text{if } S(T) > K_2, \\ S(T) - K_1 & \text{if } K_2 \geq S(T) > K_1, \\ 0 & \text{if } S(T) < K_1. \end{cases} \end{aligned}$$

- Sketch the graphs of the payoffs of the European digital call option,  $DC(T, S, K)$ , the call spread option with  $K_1 \leq K < K_2$ , and the call spread ratio  $(C(T, S, K_1) - C(T, S, K_2)) / (K_2 - K_1)$  as functions of terminal stock price,  $S(T)$ . Observe that call spread ratio approximates the payoff of the digital call if  $K_1 = K$  and  $K_2 = K + \delta K$  with  $\delta K$  small.
- Consider the binomial model with  $T = 3$ ,  $\delta t = 1$ ,  $S_0 = 4$ ,  $u = 2$ ,  $d = 1/2$ , annually compounded risk-free interest rate  $r = 1/4$ , and stock process  $S(t) = S(t, \omega)$ . If  $K_1 = 6$  and  $K_2 = 10$ , compute the corresponding no-arbitrage call spread value,  $V(0)$ , and call spread ratio  $V(0) / (K_2 - K_1)$ . Compare your answer with the value of a digital call computed in the preceding problem.
- Repeat the preceding part when  $K_2 = 7$ . Compare your answer with the value of a digital call computed in the preceding problem.

*Application:* Delta hedging a  $T$ -maturity digital call option with strike  $K$  can be expensive in practice because of transaction costs arising in buying and selling shares of the underlying,  $S(t)$ , in order to maintain the correct hedge ratio,  $\Delta(t)$ . An alternative, potentially less expensive strategy, is to notice that the call spread ratio for strikes  $K_1 = K$  and  $K_2 = K + \delta K$  and the same maturity approximates the payoff of the digital call when  $\delta K$  is small. Thus, if you sell a digital call to a client (and so are short the digital call contract), you can set up an approximate *static hedge* by buying a call spread contract with maturity  $T$ ,  $K_1 = K$ , and  $K_2 = K + \delta K$  with  $\delta K$  small (so you are long the spread contract). The hedge is called static because once you buy the call spread at time zero, no further buying or selling of options or the underlying required to maintain the hedge.

3. Let  $Z_j$ ,  $j = 1, 2, 3, \dots$  be a sequence of independent, Bernoulli random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\mathbb{P}[Z_j = 1] = 1 - p$  and  $\mathbb{P}[Z_j = -1] = p$ , where  $0 < p < 1$ . Define a scaled random walk by setting

$$W^{(n)}(t) := \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{nt} (Z_j - \mu),$$

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when  $nt$  is an integer; if  $nt$  is not an integer,  $W^{(n)}(t)$  is defined by linear interpolation between  $W^{(n)}(s)$  and  $W^{(n)}(u)$ , where  $t \in (s, u)$  and  $ns$  is the largest integer such that  $ns < nt$  and  $nu$  is the smallest integer such that  $nt < nu$ . (See §3.2.5 in Shreve II.)

- Compute the mean  $\mu$  and variance  $\sigma^2$  of  $Z_j$ .
- Compute  $\mathbb{E}[W^{(n)}(t)]$  and  $\text{Var}[W^{(n)}(t)]$  when  $nt$  is an integer.
- Show that the distribution of the random variable  $W^{(n)}(t)$  converges to the distribution of a normal random variable with mean zero and variance  $t$ , at least when  $t$  is a rational number and we take limits as  $n \rightarrow \infty$  with  $nt$  an integer. Use the Central Limit Theorem; do *not* repeat the justification in Shreve II of Theorem 3.2.1.

4. Consider the binomial model of Problem 1 in Assignment 4, where

$$S^{(n)}(t_j) = S^{(n)}(t_{j-1}) \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \delta t + \sigma \sqrt{\delta t} Z_j \right),$$

where  $S^{(n)}(0) = S(0)$ ,  $r > 0$  is the continuously compounded risk-free interest rate,  $0 < \sigma < 2/\sqrt{\delta t}$ , with  $\delta t = T/n$ , and  $Z_j$  for  $j = 1, 2, \dots, n$  is a sequence of independent Bernoulli random variables on  $(\Omega, \mathcal{F}, \mathbb{Q})$  with  $\mathbb{Q}[Z_j = 1] = 1 - q$ ,  $\mathbb{Q}[Z_j = -1] = q$  with  $0 < q < 1$  defined as in the cited problem.

- Show that

$$S^{(n)}(T) = S(0) \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{\frac{T}{n}} \sum_{j=1}^n Z_j \right).$$

- Show that the distribution of  $S^{(n)}(T)$  converges to the distribution of

$$S(T) := S(0) \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma W(T) \right),$$

where  $W(T)$  is a normal random variable with mean zero and variance  $T$ . [Hint: It suffices to consider convergence of the distribution of  $\log S^{(n)}(T)$ .]

5. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the probability space for the binomial model, with filtration  $\{\mathcal{F}_n\}_{n=0}^\infty$ , as defined in Example 1.1.4 in Shreve II. Let  $\{X_n\}_{n=1}^\infty$  be a sequence of identically distributed random variables on  $(\Omega, \mathcal{F})$  with  $X(\omega) = \pm 1$  if  $\omega = (\xi_1, \dots, \xi_n, \dots)$  and  $\xi = \pm 1$ , where  $\mathbb{P}\{X_n = -1\} = p$ ,  $\mathbb{P}\{X_n = 1\} = 1 - p$ , and  $p = 1/2$ . Define the discrete-time stochastic process  $\{M_n\}_{n=0}^\infty$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  (symmetric random walk, as in §3.2.1 in Shreve II) by setting

$$M_0 = 0 \quad \text{and} \quad M_n = \sum_{j=1}^n X_j, \quad n \geq 1.$$

- Show that  $\{M_n\}_{n=0}^\infty$  is a martingale.
- Show that  $\{M_n\}_{n=0}^\infty$  is Markov.
- Can you give an example of a discrete time martingale process which is not Markov?
- Can you give an example of a discrete time Markov process which is not a martingale?

6 (Optional). Consider the  $n$ -period binomial model with stock,  $S(t)$ , continuously compounded risk-free interest rate  $r$ ,  $S(t + \delta t) = uS(t)$  or  $dS(t)$  with risk-neutral probabilities  $1 - q$  and  $q$ ,  $\delta t = T/n$ ,  $t = j\delta t$ ,  $j = 0, 1, \dots, n$ , and market state space  $\Omega$ . Let  $V(T)$  be the payoff of an option contingent on  $S$ . In Problem 5(e) of Assignment 3 you showed that the no-arbitrage price at time  $t = 0$  of the option is

$$V(0) = D(0, T) \mathbb{E}_{\mathbb{Q}}[V(T)],$$

where  $D(0, T) = (e^{-r\delta t})^n = e^{-rT}$  (continuous compounding) has replaced  $(1 + r\delta t)^{-n}$  (discrete compounding) as the discount factor for the period  $[0, T]$ , and where  $(\Omega, \mathcal{F}, \mathbb{Q})$  has its usual meaning.

- Assume  $T = 3$  and  $\delta t = 1$ . Show that the no-arbitrage price of the option  $V(t)$  at time  $0 \leq t \leq T$  is

$$V(t) = D(t, T) \mathbb{E}_{\mathbb{Q}}[V(T) | \mathcal{F}(t)],$$

where  $\mathcal{F}(t)$  has its usual meaning and  $D(t, T) = e^{-r(T-t)}$  is the discount factor for the period  $[t, T]$ .

- (b) Use induction to generalize your argument in the preceding section and justify the formula for  $V(t)$  for arbitrary  $0 \leq t \leq T$ .

**7 (Optional).** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the probability space defined by  $\Omega = \{(\xi_1, \xi_2, \dots) : \xi_i = +1 \text{ or } -1\}$ , let  $\mathcal{F}$  be the  $\sigma$ -algebra of all subsets of  $\Omega$ , and let  $\{\mathcal{F}_n\}_{n=0}^\infty$  be the filtration of  $\mathcal{F}$  such that each  $\sigma$ -algebra  $\mathcal{F}_n$  is generated by the *atoms*, that is, subsets  $A_{\xi_1 \xi_2 \dots \xi_n} \subset \Omega$  of the form

$$A_{\xi_1 \xi_2 \dots \xi_n} := \{(\xi_1, \xi_2, \dots, \xi_n, \eta_{n+1}, \dots) \in \Omega : \eta_j = \pm 1, j \geq n+1\},$$

for each  $n \geq 1$  and fixed vector  $(\xi_1, \xi_2, \dots, \xi_n)$ . Let  $\mathbb{P}$  be the probability measure on  $\mathcal{F}_n$  defined by  $\mathbb{P}(A_{\xi_1 \xi_2 \dots \xi_n}) = (1-p)^{n-k} p^k$ , where  $k$  is the number of  $\xi_i = -1$  in  $(\xi_1, \xi_2, \dots, \xi_n)$ . Let  $X_j$ ,  $j = 1, 2, \dots$ , be a sequence of independent Bernoulli random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $X_j = 1$  or  $-1$ , with probability  $1-p$  or  $p$ . Let  $M_k := \sum_{j=1}^k X_j$  be the *random walk*, let  $p = \frac{1}{2}$ , and define the *scaled random walk*  $W^n(t)$  by setting  $W^n(t) := M_{nt}$  when  $nt$  is an integer and otherwise using linear interpolation between  $M_{ns}$  and  $M_{nu}$  for nearest integer neighbors  $ns < nt < nu$ . Show that

- $W^{(n)}(t)$  has independent increments,
- $\mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] = 0$ ,
- $\text{Var}[W^{(n)}(t) - W^{(n)}(s)] = |t - s|$ ,
- $W^{(n)}$  has quadratic variation  $t$ :  $[W^{(n)}, W^{(n)}](t) = t$ ,
- $W^{(n)}(t)$  is a martingale:  $\mathbb{E}[W^{(n)}(t) | \mathcal{F}(s)] = W^{(n)}(s)$  for all  $0 \leq s \leq t < \infty$ .

[Hint: See Shreve II §3.2.5]

**8 (Optional).** Let  $f : [0, T] \rightarrow \mathbb{R}$  be a function which is continuous on  $[0, T]$  and is differentiable on  $(0, T)$ . Let  $W(t)$  be Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Show that

- $[f, f](T) = 0$ ,
- $[W, f](T) = 0$ .

Applying the preceding two results to the function  $f(t) = t$  yields the identities informally written in differential forms as  $dt \cdot dt = 0$  and  $dW \cdot dt = 0$ , just as the identity  $[W, W](t) = t$  is often written informally in differential form as  $dW \cdot dW = dt$ . [Hints: Use the Mean Value Theorem from Calculus: On any interval  $[t_0, t_1] \subset [0, T]$  there is a  $t_0^* \in (t_0, t_1)$  such that  $f(t_1) - f(t_0) = f'(t_0^*)(t_1 - t_0)$ . See Shreve §3.4.2 and Remark 3.4.5]

**9 (Optional).** Let  $Z$  be a normal random variable with mean zero. Show that  $\mathbb{E}[Z^4] = 3\text{Var}[Z]^2$ . [Hint: See Exercise 3.3 in Shreve II.]

**10 (Optional).** Let  $W(t)$  be Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , let  $\mu$  and  $\sigma$  be constants, and let  $S(t) = S(0) \exp(\mu t + \sigma W(t))$  be geometric Brownian motion. Show that the process  $\log S(t)$  has quadratic variation on the interval  $[0, T]$  given by

$$[\log S, \log S](T) = \sigma^2 T.$$

[Hint: See Shreve §3.4.3.]

**11 (Optional).** *Central Limit Theorem* [Chung (1974), Theorem 6.4.4]. Let  $\{X_n\}_{n=1}^\infty$  be a sequence of independent, identically distributed random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . If  $\mathbb{E}[X_n] = \mu$  and  $\text{Var}[X_n] = \sigma^2$ , then

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{1}{\sigma \sqrt{n}} \sum_{j=1}^n (X_j - \mu) \leq x \right\} = \Phi(x), \quad x \in \mathbb{R},$$

where  $\Phi$  is the normal distribution function with mean zero and variance one. Prove this result by adapting the justification of Theorem 3.2.1 in Shreve II.