MATH FINANCE I
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Note Title 9/11/2007

Binomial Branch model

- S. Pliska, "Discrete-time financial models" (1997)
- S. Shreve, "Discrete time financial models"

Two assets: $S_1(t,\omega) = B(t)$, $S_2(t,\omega) = S(t,\omega)$

$B(0) = B_0$, $B(T) = (1+rT)B_0$ (simple interest)

Two market states: $\Omega = \{\omega_1, \omega_2\} = \{u, d\}$

One-period, $S(T,\omega_1) = uS_0$, $S(T,\omega_2) = dS_0$

Assume $B_0 > 0$, $S_0 > 0$.

Suppose $X(\omega)$ is a "contingent claim" (or "option" or "derivative security");
$X : \Omega \to \mathbb{R}$, $X(\omega_1), X(\omega_2)$ known.

Let $\Phi = \{\phi_1, \phi_2\}$ is a portfolio with value

$$\begin{aligned} V(T,\omega) &= \Phi A(T,\omega), \quad \omega \in \Omega \\ &= \phi_1 S_1(T,\omega) + \phi_2 S_2(T,\omega) \\ &= \phi_1 B(T) + \phi_2 S(T,\omega) \end{aligned}$$

Seek a replicating portfolio for X: Φ s.t.

$$V(T,\omega) = X(\omega) \text{ for all } \omega \in \Omega$$

Using $\omega = u, d$ we find that

$$V(T,u) = \phi_1 B(T) + \phi_2 S(T,u)$$
$$V(T,d) = \phi_1 B(T) + \phi_2 S(T,d)$$

$\Rightarrow$ Solve for ϕ_1, ϕ_2 as in HW #1.

Assuming $d < 1+rT < u$, so that market is arbitrage-free.

Principle of no arbitrage $\Rightarrow$

$$V(0) = \phi_1 B(0) + \phi_2 S(0)$$

where $V(0)$ is the "fair" or no arbitrage price of X in market $\{B, S, V\}$.

Simplify, to find that

$$\boxed{V(0) = \frac{D(T)}{D(0)} \{ q_d V(T,d) + q_u V(T,u) \}}$$

"Risk-neutral" pricing formula,

$D(T) = 1/B(T)$, $D(0) = 1/B(0)$, and

$$q_d = \frac{u-(1+rT)}{u-d}, \quad q_u = \frac{(1+rT)-d}{u-d}$$

Note: $q_u > 0$, $q_d > 0$ and $q_u + q_d = 1$

So usually choose $q_u = q$, so $q_d = 1-q$.

State Price Vector, Risk Neutral "Measure" and Risk-Neutral (or No Arbitrage) Price

Defn: Consider 1-period model, p assets, m states, $\Omega = \{\omega_1, \dots, \omega_m\}$, price functions $\{S_1(t,\omega), \dots, S_p(t,\omega)\}$, $t = 0$ or T.

Asset Price Vector: $A(t,\omega) = \begin{pmatrix} S_1(t,\omega) \\ \vdots \\ S_p(t,\omega) \end{pmatrix}_{p\times 1}$

matrix of prices: $\mathcal{A} = \begin{pmatrix} S_1(T,\omega_1) & \cdots & S_1(T,\omega_m) \\ \vdots & & \vdots \\ S_p(T,\omega_1) & \cdots & S_p(T,\omega_m) \end{pmatrix}_{p\times m}$

$A(0) = \begin{pmatrix} S_1(0) \\ \vdots \\ S_p(0) \end{pmatrix}_{p\times 1}$

We call $\psi = (\psi_1, \dots, \psi_m)^* \in \mathbb{R}^m$ is a state price vector for the given market if

$\psi_j > 0$ for $j = 1, \dots, m$ and

$$\boxed{\underset{p\times 1}{A(0)} = \underset{p\times m}{\mathcal{A}}\ \underset{m\times 1}{\psi}}$$

Define $\Pi(t,\omega) := \underset{1\times p}{\Phi}\ \underset{p\times 1}{A(t,\omega)}$. Then Φ is an arbitrage if

(a) $\Pi(0) < 0$ and $\Pi(T,\omega) \geq 0$ $\forall \omega \in \Omega$ or

(b) $\Pi(0) = 0$ and $\Pi(T, \omega) \geq 0$ $\forall\, \omega \in \Omega$ and
$\Pi(T, \omega') > 0$ some $\omega' \in \Omega$

Market is arbitrage-free if $\nexists$ an arbitrage portofio (trading strategy.

Theorem: The 1-period, finite-state model is arbitrage free $\iff$ there is a state price vector for this model.

First Fundamental Theorem of Asset Pricing

Interpretation of the State Price Vector

Suppose $\exists\ \Phi = (\phi_1, \dots, \phi_p)$ such that

$$\Pi(T,\omega) := \Phi \cdot A(T, \omega_i) = 1, \qquad i = 1, \dots, m$$

(Like a "zero-coupon bond".)

Such a portfolio is risk-free: independent of market state $\omega_i \in \Omega$ at T.

$$\Rightarrow \Pi(0) = \Phi \cdot A(0) = \sum_{i=1}^{p} \phi_i S_i(0)$$

$$\Rightarrow \text{Risk-free return} = \frac{1}{\Phi \cdot A(0)}$$

$$\Pi(0) = \Phi \cdot A(0)$$
$$\Pi(T, \omega) = 1 \qquad \forall\, \omega \in \Omega$$

Example (Exercise) If $S_1(T,\omega) = B(T)$ (money market account) then

$$B(T) = \frac{1}{\Phi \cdot A(0)}$$ and $D(T) = \Phi \cdot A(0)$ is called the discount factor.

Example (Exercise) $\Phi \cdot A(0) = \sum_{i=1}^{m} \psi_i$, and positive.

Soln: $\underbrace{\Phi \cdot A(0)}_{} = \underbrace{\Phi A \psi}_{} = (\Phi A) \cdot \psi = (1, \dots, 1) \cdot \psi$

$$= \sum_{i=1}^{m} \psi_i$$

using $\underset{1 \times p}{\Phi} \underset{p \times 1}{A(T, \omega_i)} = 1$, $i = 1, \dots, m$

ie $\underset{1 \times p}{\Phi} \underset{p \times m}{A} = \underbrace{(1, \dots, 1)}_{m}$. $\square$

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State Price Vector (continued)

Completeness: Given a derivative security with payoff $X : \Omega \to \mathbb{R}$, ie $X(\omega)$, $\omega \in \Omega$, then the market is complete if there is a replicating portfolio, $\Phi = (\phi_1, \dots, \phi_p)$:

$$X(\omega) = \sum_{j=1}^{p} \phi_j S_j(T, \omega) \quad \text{for all } \omega \in \Omega$$

$=: V(T, \omega)$ (portfolio)

then $V(T) = \left(V(T,\omega_1), \ldots, V(T,\omega_m)\right)_{1\times m}$

$$= (\phi_1, \ldots, \phi_p) \begin{pmatrix} S_1(T,\omega_1) & \cdots & S_1(T,\omega_m) \\ \vdots & & \vdots \\ S_p(T,\omega_1) & \cdots & S_p(T,\omega_m) \end{pmatrix}$$

$$\underset{1\times m}{V(T)} = \underset{1\times p}{\Phi} \underset{p\times m}{A(T)}$$

Suppose market is arbitrage free, so ψ exists so that $\underset{p\times 1}{A(0)} = \underset{p\times m}{A}\underset{m\times 1}{\psi}$.

Arbitrage-freeness $\Rightarrow V(0) = \underset{1\times p}{\Phi} \cdot \underset{p\times 1}{A(0)}$

where $V(0)$ is no-arbitrage price at $t=0$ of contract with payoff $X(\omega) \equiv V(T,\omega)$.

ψ exists $\Rightarrow$ $V(0) = \underset{1\times p}{\Phi}(\underset{p\times m}{A}\underset{m\times 1}{\psi}) = (\Phi A)\psi$

$$= \underset{1\times m}{V(T)}\underset{m\times 1}{\psi}$$

$$= \left(V(T,\omega_1), \ldots, V(T,\omega_m)\right)\begin{pmatrix}\psi_1 \\ \vdots \\ \psi_m\end{pmatrix}$$

$$\Rightarrow \boxed{V(0) = \sum_{i=1}^{m} \psi_i V(T,\omega_i)}$$

Risk-neutral pricing formula in terms of the state price vector.

Risk-return: $D(T) = \sum_{i=1}^{m} \psi_i$

$$\Rightarrow \boxed{V(0) = D(T) \cdot \sum_{i=1}^{m} q_i V(T, \omega_i)}, \quad \text{where}$$

$$q_i = \frac{\psi_i}{\sum_{j=1}^{m} \psi_j}$$

Notice: $q_1 + \cdots + q_m = 1$, $q_i > 0$, $i = 1, \ldots, m$

Define $\mathbb{Q} : \Omega \to [0,1]$, <u>risk-neutral measure</u>, by

$$\mathbb{Q}(\omega_i) = q_i$$

Then $V(0) = D(T) \sum_{i=1}^{m} \mathbb{Q}(\omega_i) V(T, \omega_i)$

<u>ie</u> $$\boxed{V(0) = D(T)\, \mathbb{E}_{\mathbb{Q}}[V(T)]}$$

<u>Arrow-Debreu Security</u> The ith Arrow-Debreu security has <u>payoff</u>

$$e_i(\omega) = \begin{cases} 1, & \omega = \omega_i \\ 0, & \omega \neq \omega_i \end{cases}$$

Suppose the market is arbitrage free and complete: can find a state price vector ψ (arbitrage-freeness)

$$\Rightarrow V(0) = \sum_{k=1}^{m} \psi_k V(T, \omega_k)$$

where $V(T, \omega)$ = value of portfolio replicating i^{th} Arrow-Debreu security, e_i

$$\Rightarrow \quad V(0) = \psi_i V(T, \omega_i) = \psi_i \underbrace{e_i(\omega_i)}_{=1} = \psi_i$$

because $\quad V(T, \omega_k) = e_i(\omega_k) = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$

$\Rightarrow$ $\boxed{\psi_i \text{ is the no arbitrage price at } t=0 \text{ of the } i^{th} \text{ Arrow-Debreu security, } e_i}$

Key Ideas in the Proof of the First Fundamental Theorem of Asset Pricing

Model is arbitrage free $\iff$ $\exists\ \psi = (\psi_1, \dots, \psi_m)$ so that $\underset{p \times 1}{A(0)} = \underset{p \times m}{A}\ \underset{m \times 1}{\psi}$

and $\psi_i > 0, \quad i = 1, \dots, m$

Separating Hyperplane Theorem

Theorem: Let $\mathbb{K}$, $\mathbb{M}$ be two closed, convex cones in $\mathbb{R}^{m+1}$. Then

$$\begin{cases} \text{(i) } \mathbb{K} \cap \mathbb{M} = \{\vec{0}\} \subset \mathbb{R}^{m+1} \\ \text{(ii) } \mathbb{K} \text{ contains no lines through } 0 \in \mathbb{R}^{m+1} \end{cases}$$

$\iff \exists\ \alpha$ in $\mathbb{R}^{m+1}$, $\alpha \neq \vec{0}$, so that

$$\begin{cases} \alpha \cdot x > 0, & \forall x \text{ in } \mathbb{K},\ x \neq \vec{0} \\ \alpha \cdot y \leq 0, & \forall y \text{ in } \mathbb{M} \end{cases}$$

Also see Duffie, "Dynamic Asset Pricing Theory".

Complete & Incomplete Markets & State Price Vector

$X : \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega)$ is a "contingent claim", "option", "derivative security".

Can view as random variable & represents payoff of some contract.

X is attainable in market if $\exists$ trading strategy $\Phi = (\phi_1, \ldots, \phi_p)$ so that

$$X(\omega) = \Pi(T, \omega) := \sum_{j=1}^{p} \phi_j S_j(T, \omega) \quad \forall \omega$$

Buyer of contract pays $\Pi(0)$ ($=?$) @ $t=0$ to seller of contract paying $X(\omega)$ to buyer @ $t=T$.

Principle of no arbitrage $\Rightarrow$

$$\Pi(0) = \sum_{j=1}^{p} \phi_j S_j(0)$$

Second Fundamental Theorem of Asset Pricing Theory

Assume model is arbitrage-free. Then model is complete $\Leftrightarrow$ the risk-neutral measure Q (ie the state price vector ψ) is unique.

Recall "First" theorem: model is arbitrage free $\Leftrightarrow$ risk-neutral measure $\mathbb{Q}$ (ie state price vector ψ) exists.

Examples ① $r = 1/9$, $S_1 = B$, $S_2 = S$ with prices

$B(0) = 1$, $B(T) = (1+r)B(0) = 1 + 1/9 = 10/9$

$S(0) = 5$, $S(T,\omega_1) = 20/3$, $S(T,\omega_2) = 40/9$

$\Omega = \{\omega_1, \omega_2\}$

Is arbitrage-free market?

Can we find ψ (or $\mathbb{Q}$) so that

$$\boxed{\begin{array}{l} q_1 + q_2 = 1, \quad q_1 > 0, \ q_2 > 0 \quad \text{and} \\ S_j(0) = \mathbb{E}_{\mathbb{Q}}[S_j(T)]\, D(T), \ j = 1, 2 \end{array}}$$

$A(0) = A\psi$ (rewritten)

$j=1$: $1 = S_1(0) = B(0) = \mathbb{E}_{\mathbb{Q}}[S_1(T)]\, D(T)$

$$= \mathbb{E}_{\mathbb{Q}}[B(T)]\, D(T) = B(T)\, D(T)$$

$\Rightarrow$ $D(T) = 1/B(T) = 9/10$

$j=2$: $S_2(0) = S(0) = \mathbb{E}_{\mathbb{Q}}[S_2(T)]\, D(T)$

ie $5 = (q_1 S(T,\omega_1) + q_2 S(T,\omega_2)) \frac{9}{10}$

ie $5 = \left(q_1 \cdot \frac{20}{3} + q_2 \cdot \frac{40}{9}\right) \cdot \frac{9}{10}$

ie $\boxed{5 = 6q_1 + 4q_2}$ and $\boxed{\begin{matrix} q_1 + q_2 = 1 \\ q_1 > 0,\ q_2 > 0 \end{matrix}}$

Suppose $q_1 = q_2 = \frac{1}{2}$?

Works! Arbitrage-free model, because $\mathbb{Q}\,(\equiv \psi)$ exists.

Complete model, because $\mathbb{Q}\,(\equiv \psi)$ unique $\square$

<u>Example</u> $m = 3$, $p = 2$ with $S_1 = B$, $S_2 = S$

$B(0) = 1$, $r = 1/9$, $B(T) = \frac{10}{9}$ and $S(0) = 5$,

$$S(T, \omega_1) = \frac{20}{3}, \quad S(T, \omega_2) = \frac{40}{9}, \quad S(T, \omega_3) = \frac{30}{9}$$

model arbitrage-free?

<u>Solution</u>: check existence of ψ so that

$A(0) = A\psi$ (ie $\mathbb{Q}$ so that)

ie $$S_j(0) = \sum_{i=1}^{m} \psi_i\, S_j(T, \omega_i)$$

ie $$S_j(0) = D(T) \sum_{i=1}^{m} q_i\, S_j(T, \omega_i)$$

where $q_i = \frac{\psi_i}{D(T)}$, $D(T) = \sum_{i=1}^{m} \psi_i$.

ie $$S_j(0) = D(T)\, \mathbb{E}_{\mathbb{Q}}[S_j(T)]$$

<u>Solution: $j=1$</u>, $D(T) = 1/B(T) = 9/10$

<u>$j=2$</u>: $S(0) = D(T) \{ q_1 S(T, \omega_1) + q_2 S(T, \omega_2) + q_3 S(T, \omega_3) \}$

$$\Rightarrow \quad 5 = \frac{9}{10}\left\{ q_1 \cdot \frac{20}{3} + q_2 \frac{40}{9} + q_3 \cdot \frac{30}{9} \right\}$$

$$\Rightarrow \quad \boxed{5 = 6q_1 + 4q_2 + 3q_3} \qquad \boxed{\begin{array}{l} q_1 + q_2 + q_3 = 1 \\ q_1 > 0,\ q_2 > 0\ q_3 > 0 \end{array}}$$

Find $(q_1, q_2, q_3) = (\lambda, 2-3\lambda, -1+2\lambda)$, $\lambda \in \mathbb{R}$

where $\frac{1}{2} < \lambda < \frac{2}{3}$.

Infinitely many solutions exist.

$\Rightarrow$ model is arbitrage-free.

Is model complete?

2nd Fund theorem says market cannot be complete. Geometrically ...

$$A = \begin{pmatrix} S_1(T,\omega_1) & S_1(T,\omega_2) & S_1(T,\omega_3) \\ S_2(T,\omega_1) & S_2(T,\omega_2) & S_2(T,\omega_3) \end{pmatrix}$$

$$= \begin{pmatrix} 10/9 & 10/9 & 10/9 \\ 20/3 & 40/9 & 30/9 \end{pmatrix}$$

Market is complete $\Leftrightarrow$ solution φ to $\underset{(p\times 1)}{A(\omega)} = \underset{(p\times m)(m\times 1)}{A\varphi}$ is <u>unique</u>

φ (or $\mathbb{Q}$) not unique, so not complete.

For example, spse $X(\omega) = (X(\omega_1), X(\omega_2), X(\omega_3))$. Can we find Φ so that

$\Phi A(T,\omega) = X(\omega)$, $\omega \in \Omega$

i.e. $\Phi A(T,\omega_1) = X(\omega_1)$, $\Phi = (\phi_1, \phi_2)$

$\Phi A(T,\omega_2) = X(\omega_2)$, $A(T,\omega) = \begin{pmatrix} B(T) \\ S(T,\omega) \end{pmatrix}$

$\Phi A(T,\omega_3) = X(\omega_3)$

i.e. $\frac{10}{9}\phi_1 + \frac{20}{9}\phi_2 = X_1$

$\frac{10}{9}\phi_1 + \frac{40}{9}\phi_2 = X_2$

$\frac{10}{9}\phi_1 + \frac{30}{9}\phi_2 = X_3$

Find that solution exists, i.e. $X(\omega)$ is attainable, iff

$$\boxed{X_1 - 3X_2 + 2X_3 = 0} \quad , \quad X_i = X(\omega_i)$$

other payoffs <u>not</u> attainable + market not complete.