16:642:621 MATH FINANCE I 10/16/2007

Note Title 10/16/2007

MARKOV PROPERTY OF BROWNIAN MOTION

THM: Spse $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $W(t)$ is Brownian motion adapted to a filtration $\mathcal{F}(t)$, $t \geq 0$. Then $W(t)$ is a a Markov process.

Defn: $X(t)$ is an adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$, filtration $\mathcal{F}(t)$. Then $X(t)$ is Markov if, given a function $f: \mathbb{R} \to \mathbb{R}$, then there is a function $g: \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}[f(X(t)) \mid \mathcal{F}(s)] = g(X(s)), \quad 0 \leq s \leq t.$$

Lemma (Independence Lemma) $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \subset \mathcal{F}$. Spse X $\mathcal{G}$-measurable random variable and Y is another random variable, independent of $\mathcal{G}$. Given $f(x,y)$, define

$$g(x) := \mathbb{E}[f(x,Y)]$$

Then $\mathbb{E}[f(X,Y) \mid \mathcal{G}] = g(X)$

<u>Why is Brownian motion Markov?</u>

$$\mathbb{E}[f(W(t)) \mid \mathcal{F}(s)] = \mathbb{E}[f(\underline{W(t) - W(s)} + W(s)) \mid \mathcal{F}(s)]$$

By defn of Brownian motion, $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ & $N(0, t-s)$.

Define $g(x) := \mathbb{E}[f(W(t) - W(s) + x) \mid \mathcal{F}(s)]$

(independence) $= \mathbb{E}[f(W(t) - W(s) + x]$

$$= \int_{\mathbb{R}} f(w+x)\,\phi_{0,t-s}(w)\,dw$$

Then $g(W(s)) = \mathbb{E}\left[f(W(t)-W(s) + W(s) \mid \mathcal{F}(s) \right]$

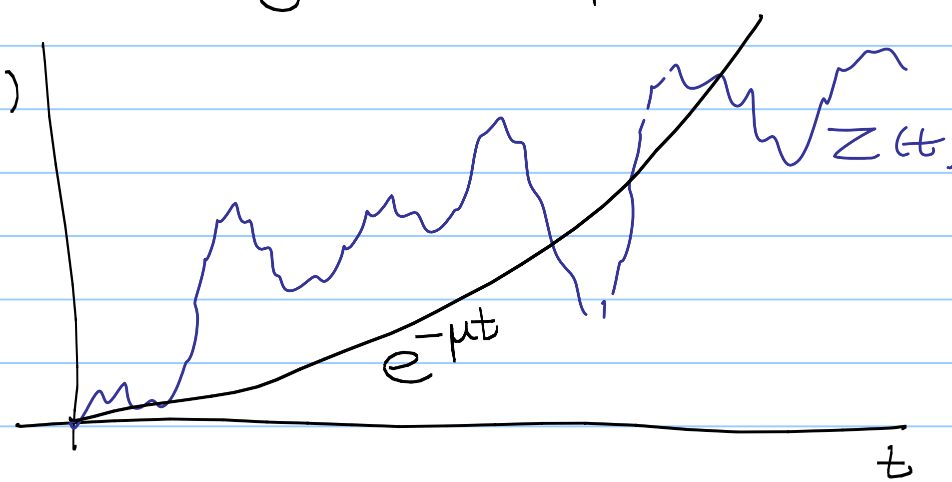
Independence Lemma. $\square$

<u>Example</u> Suppose $Z(t) = \exp(\sigma W(t) - \mu t)$ (volatility; drift)

Geometric Brownian motion $\qquad = e^{-\mu t} e^{\sigma W(t)}$.

Show that $Z(t)$ is a martingale $\iff \mu = \sigma^2/2$.

Sample paths $(\mu < 0)$:



Itô Calculus

Meaning of

$$\boxed{I(t) = \int_0^t \Delta(s)\, dW(s), \quad t \geq 0}$$

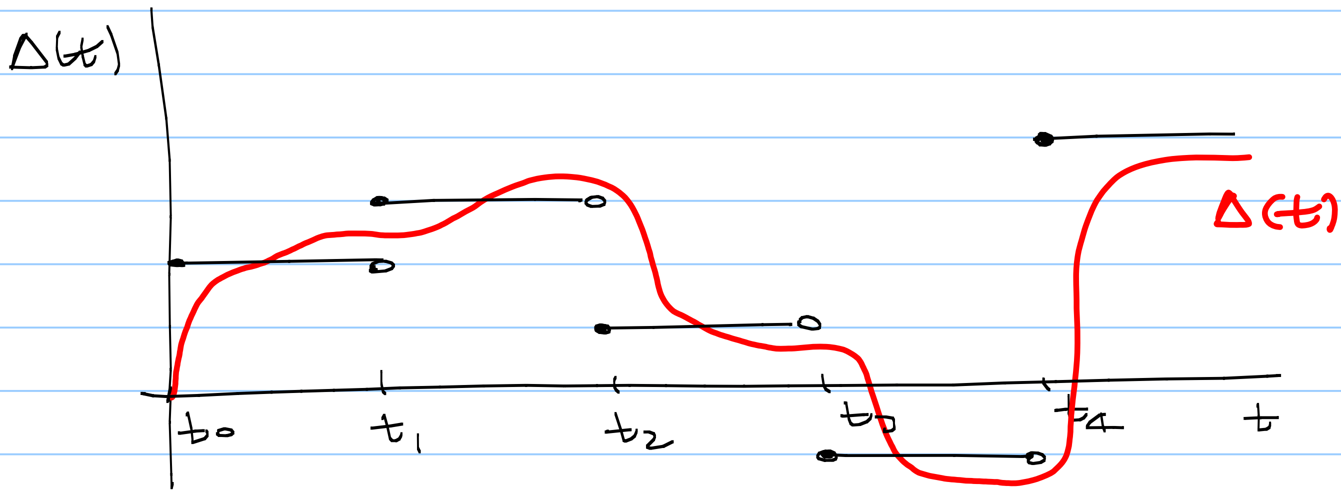
$W(t)$ is Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}(t)$ filtration for $W(t)$, $\Delta(t)$ is an adapted process.

Spse "$dW(s) = \frac{dW(s)}{ds} ds \equiv W'(s)\, ds$". Then

"$I(t) = \int_0^t \Delta(s) \frac{dW(s)}{ds} ds$". Does not make sense as $W(t)$ is not differentiable w.r.t. time t.

Construction of Itô Integral

Assume $\Delta(t)$ is a simple process:



(Piecewise-constant sample paths.)

choose $\Pi := \{t_0, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ &
consider $\Delta(s)$ constant for each $s \in [t_j, t_{j+1})$,

$j = 0, \dots, n-1$, given $\omega \in \Omega$.

Define $I(t) := \sum_{j=0}^{k-1} \Delta(t_j) [W(t_{j+1}) - W(t_j)]$
$+ \Delta(t_k) [W(t) - W(t_k)]$

where $t \in [t_k, t_{k+1})$, and denote

$$I(t) = \int_0^t \Delta(s)\, dW(s) \quad (\text{shorthand for} \quad)$$

Called the Itô integral of $\Delta(t)$ w.r.t. $W(t)$.

<u>Abbreviations</u>: "$dI(t) = \Delta(t)\, dW(t)$"

means $I(t) - I(0) = \int_0^t \Delta(s)\, dW(s)$ (if $I(0) \neq 0$).

Properties of Itô Integrals

Thm: $I(t)$ is a martingale process

thm: $\mathbb{E}[I(t)] = 0$ (if $I(0)=0$) and

$\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t \Delta(u)^2\,du\right]$ (function of t)

where "$\int_0^t \Delta(u)^2\,du$" defined by $\int_0^t \Delta(u,\omega)^2\,du$ $\forall\, \omega \in \Omega$.

Thm $[I,I](t) = \int_0^t \Delta^2(u)\,du$ (stochastic process)

(Quadratic variation)

Abbreviation: $dI(t)\cdot dI(t) = \Delta^2(t)\,dt$

Construction of the Itô Integral for General Integrands

$$I(t) = \int_0^t \Delta(s)\, dW(s) \quad , \quad t \geq 0.$$

- $\Delta(t)$ is $\mathcal{F}(t)$-adapted
- $\mathbb{E}\left[\int_0^t \Delta(s)^2\, ds\right] < \infty$

Idea: Approximate $\Delta(t)$ by a sequence of simple processes $\Delta_n(t)$ (for given Π)

Take limits as $\|\Pi\| \to 0$, $n \to \infty$ and define

$$I(t) := \lim_{n\to\infty,\, \|\Pi\|\to 0} \int_0^t \Delta_n(s)\, dW(s)$$

where $\Delta_n(t) \longrightarrow \Delta(t)$ converges in sense that

$$\lim_{n\to\infty} \mathbb{E}\int_0^t |\Delta(s) - \Delta_n(s)|^2 ds \longrightarrow 0$$

Properties of the Itô Integral

Fix $T > 0$, let $\Delta(t)$, $0 \le t \le T$, be an adapted process such that

$$\mathbb{E}\int_0^T \Delta^2(u)\,du < \infty.$$

Then

$$I(t) := \int_0^t \Delta(u)\,dW(u)$$

has the following properties:

(1) Continuity: Sample paths $I(t,\omega)$, $\omega \in \Omega$, of $I(t)$ are continuous functions of $t \ge 0$, any $\omega \in \Omega$.

(2) Adapted: $I(t)$ is $\mathcal{F}(t)$-measurable.

(3) Martingale: $I(t)$ is a martingale.

(4) Itô Isometry: $\mathbb{E}[I(t)^2] = \mathbb{E}\left[\int_0^t \Delta(u)^2\,du\right]$

(5) Quadratic Variation: $[I,I](t) = \int_0^t \Delta(u)^2\,du$

Example: $\int_0^t W(u)\,dW(u) = \frac{1}{2}W(t)^2 - \frac{1}{2}t$

Compare: $\int_0^t g(u)\,dg(u)$ (for $g'(u)$ continuous, $g(0)=0$)

$= \int_0^t g(u)\,g'(u)\,du$

$= \frac{1}{2}g(u)^2\Big]_0^t = \frac{1}{2}g(t)^2$

(chain rule in reverse)

<u>Example</u> $\frac{1}{2}W(t)^2 - \frac{t}{2}$ is a martingale.

(because an Itô integral). Show directly.

<u>Itô Formula for Brownian Motion</u>

<u>Thm</u> Suppose $f : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$, $(t,x) \longmapsto f(t,x)$ is a $C^{1,2}$ function (f_t continuous, f_{xt}, f_{xx} continuous). Then

$$f(T, W(T)) - f(0, W(0))$$

$$= \int_0^T f_t(t, W(t)) \qquad + \int_0^T f_x(t, W(t))$$

$$+ \frac{1}{2}\int_0^T f_{xx}(t, W(t)) \qquad \text{(Integral Version)}$$

or

$$df(t, W(t)) = f_t(t, W(t)) \qquad + f_x(t, W(t))$$

$$+ \frac{1}{2} f_{xx}(t, W(t)) \qquad \text{(Differential form)}$$

Recall <u>Taylor's</u> <u>Theorem</u>:

$$f(t+dt, x+dx)$$

$$= f(t,x) + f_t(t,x)\,dt + f_x(t,x)\,dx + \frac{1}{2}\left\{ f_{tt}(t,x)(dt)^2 + 2f_{tx}(t,x)\,dt\,dx + f_{xx}(t,x)(dx)^2 \right\} + \cdots$$

In regular calculus: $(dt)^2 = (dx)^2 = dt\,dx = 0$.

$$\underbrace{f(t+dt, x+dx) - f(t,x)}_{=:\, df(t,x)} = f_t(t,x)\,dt + f_x(t,x)\,dx$$

In Itô calculus: $(dt)^2 = dt \cdot dW(t) = 0$ but

$$dW(t)^2 = dt \neq 0.$$

$\Rightarrow$ (when replacing x by $W(t)$ above):

$$df(t, W(t)) = f_t(t, W(t))\,dt + f_x(t, W(t))\,dW(t) + \frac{1}{2} f_{xx}(t, W(t))\,\underbrace{dW(t)^2}_{=dt}$$

<u>Itô Processes</u> $W(t)$ is Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to $\mathcal{F}(t)$.

An <u>Itô process</u> is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Theta(u) \qquad + \int_0^t \Delta(u)$$

where $X(0)$ is non-random, $\Theta(u)$ & $\Delta(u)$ are adapted processes. More usually:

$$dX(t) = \Theta(t)\,dt + \Delta(t)\,dW(t)$$

Technical conditions: $\cdot\ \mathbb{E}\left[\int_0^t \Delta(u)^2 du\right] < \infty,$

$\cdot\ \mathbb{E}\left[\int_0^t |\Theta(u)| du\right] < \infty, \quad \forall t > 0$

Motivation: Idea is that $f(t, W(t))$ is an Itô process.

Compare: $dX(t) = \Theta(t)\, dt + \Delta(t)\, dW(t)$

$$df(t, W(t)) = \left\{ f_t + \frac{1}{2} f_{xx} \right\} dt + f_x\, dW(t)$$

Lemma Quadratic Variation of the Itô process:

$$[X, X](t) = \int_0^t \Delta(u)^2 du$$

ie $dX(t) \cdot dX(t) = \Delta(t)^2 dt$

ie $$(dX(t))^2 = \Delta(t)^2\,dt$$

Why is formula true? Explanation:

$$(dX(t))^2 = \left(\Theta(t)\,dt + \Delta(t)\,dW(t)\right)^2$$

$$= \Theta(t)^2 (dt)^2 + 2\,\Theta(t)\,\Delta(t)\,dt \cdot dW(t) + \Delta(t)^2 (dW(t))^2$$

$$= \Delta(t)^2 (dW(t))^2 = \Delta(t)^2\,dt$$

using $(dt)^2 = dt \cdot dW(t) = 0$.

More general Itô Integrals

We know $\int_0^t \Delta(u)\,dX(u)$ when $X(t) = W(t)$.

What about $X(t) =$ Itô process?

Defn: $X(t)$ is an Itô process, $\Gamma(t)$ is an adapted process. Then

$$\int_0^t \Gamma(u)\, dX(u) := \int_0^t \Gamma(u)\,\Theta(u)\, du + \int_0^t \Gamma(u)\Delta(u)\, dW(u)$$

where $dX(t) = \Theta(t)\, dt + \Delta(t)\, dW(t)$.

Thm Itô Formula for an Itô Process

$X(t)$ is an Itô process, f is $C^{1,2}$ on $[0,\infty)\times\mathbb{R}$. Then $\forall\, T \geq 0$ we have

$$f(T, X(T)) - f(0, X(0))$$

$$= \int_0^T f_t(t, X(t))\, dt + \int_0^T f_x(t, X(t))\, dX(t)$$

$$+ \frac{1}{2}\int_0^T f_{xx}(t, X(t))\, d[X,X](t)$$

(Integral Version)

where: $dX(t) = \Theta(t)\,dt + \Delta(t)\,dW(t)$

$$d[X,X](t) = \Delta(t)^2\,dt = (dX(t))^2$$

$$df(t,X(t)) = f_t(t,X(t))\,dt + f_x(t,X(t))\,\text{[illegible]}$$

$$+ \frac{1}{2} f_{xx}(t,X(t))\; dX(t)^2$$

exercise

$$= \left(f_t + f_x\,\Theta + \frac{1}{2} f_{xx}\,\Delta^2\right) dt + f_x\,dW(t)$$

<u>Example</u> Spse $X(t)$ is defined by

$$dX(t) = \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right) dt + \sigma(t)\,dW(t)$$

and $S(t) = f(t,X(t)) := S_0\,e^{X(t)}$.

What is $dS(t)$? $\alpha(t)$, $\sigma(t)$ adapted processes.

Soln: Let $f(x) = S_0 e^{x} = f(t,x)$ (constant wrt time t)

Then $S(t) = f(X(t))$. Itô's formula:

$$dS(t) = df(X(t))$$

$$= f_t(X(t))\,dt + f_x(X(t))\,dX(t) + \frac{1}{2} f_{xx}(X(t))\cdot(dX(t))^2$$

But $f_x = S_0 e^{x} = f$, $f_{xx} = S_0 e^{x} = f$. Then:

d [redacted] $= f(X(t))\left(dX(t) + \frac{1}{2}(dX)^2 \right)$

$$= S(t)\left(\left(\alpha(t) - \frac{\sigma^2(t)}{2}\right) dt + \sigma(t)\,dW(t) + \frac{1}{2}\sigma(t)^2 \underbrace{dW(t)^2}_{= dt} \right)$$

$=$ [redacted] $\left(\alpha(t)\,dt + \sigma(t)\,dW(t)\right)$

i.e. $$\frac{dS(t)}{S(t)} = \alpha(t)\,dt + \sigma(t)\,dW(t)$$

volatility coefficient

drift coefficient

(Stochastic differential equation or SDE defining <u>Brownian</u> <u>motion</u>.)

<u>Generalized</u> <u>Geometric</u>

<u>Example</u> (Vasicek Interest Rate Model)

$R(t)$ is an interest rate process defined by

$$d\,[\text{illegible}] = (\alpha - \beta\,[\text{illegible}])\,dt + \sigma\,dW(t)$$

where α, β, σ are positive constants.

What is $R(t)$? Show

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW(s)$$

See Look for one approach.

Another approach: Use integrating factor.

$$d(\mu(t)R(t)) = \mu(t)\,dR(t) + d\mu(t)\cdot dR(t)$$

$$+ \mu(t)\left[(\alpha-\beta R(t))dt + \sigma\, dW(t)\right]$$

$$+ (d\mu(t))\, R(t)$$

(Justify by $f(t, R(t)) := \mu(t)R(t)$, where $\mu(t)$ is deterministic function)

$$= \underbrace{(d\mu(t) - \mu(t)\beta\, dt)}_{\text{(want } = 0)} R(t) + \mu(t)(\alpha\, dt + \sigma\, dW(t))$$

Choose $\mu(t)$ so that $\frac{d\mu}{dt} = \mu\beta \Rightarrow \mu(t) = e^{\beta t}$

$$\Rightarrow d(\mu(t)R(t)) = \mu(t)(\alpha\, dt + \sigma\, dW(t))$$

$$\Rightarrow \mu(t)R(t) - \mu(0)R(0) = \int_0^t \alpha e^{\beta s} ds + \int_0^t \sigma e^{\beta s} dW(s)$$

$$\Rightarrow R(t) = e^{-\beta t} R(0) + e^{-\beta t}\left\{ \frac{\alpha}{\beta}(e^{\beta t} - 1) + \int_0^t \sigma e^{\beta s} dW(s) \right\}$$

$\Rightarrow$ result. $\square$

<u>Properties</u> $R(t)$ is mean reverting

$$dR(t) = \left(\frac{\alpha}{\beta} - R(t)\right)\beta dt + \sigma dW(t) \qquad (\alpha > 0,\ \beta > 0)$$

Can show $\lim_{t \to \infty} \mathbb{E}[R(t)] = \frac{\alpha}{\beta}$

Consider $R(t) > \frac{\alpha}{\beta} \Rightarrow R(t) \downarrow \alpha/\beta$

$R(t) < \frac{\alpha}{\beta} \Rightarrow R(t) \uparrow \frac{\alpha}{\beta}$

$R(t) = \frac{\alpha}{\beta} \Rightarrow$ $R(t)$ fluctuates around long-term mean α/β.