Example Binomial tree model & computational advantages of "markov" processes.

$S(0) = S_0 = 4$, $u = 2$, $d = 1/2$, $r = 1/4$, $B_0 = 1$
$\delta t = 1$, trading times $t_n = n$, $n = 0, \dots, N$

ⓐ $N = 100$. Comment on computational aspects of computing

$$V(0) = D(T) \mathbb{E}_Q [V(T)]$$

ⓑ Compute $V(0)$ when payoff is $X(\omega) = (K - S(T,\omega))^+$, $K = 5$, $T = N = 3$.

Solution: ⓐ $V(0) = D(T) \mathbb{E}_Q [V(T)]$, $X(\omega) \equiv V(T,\omega)$

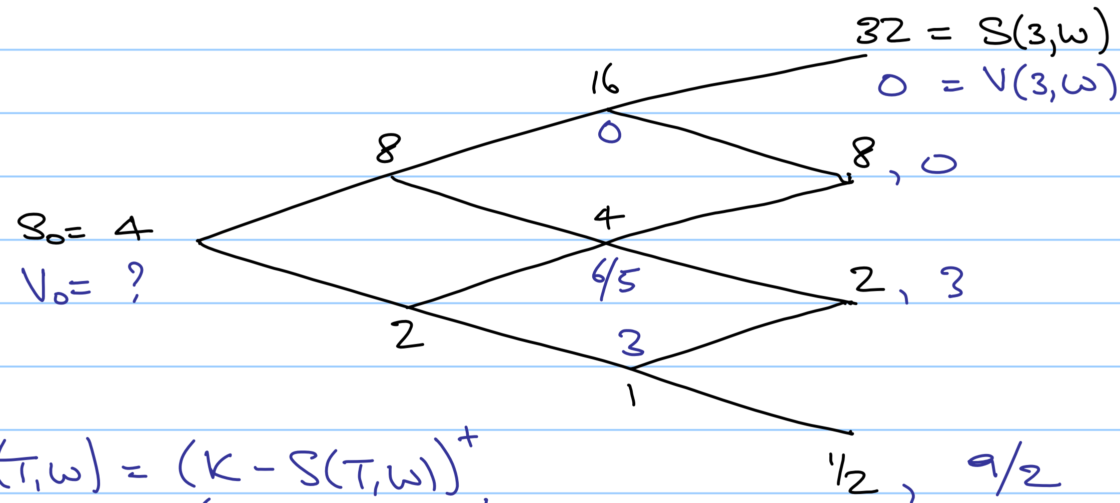
$$= \frac{1}{(1 + r\delta t)^N} \sum_{\omega \in \Omega} Q(\omega) V(T,\omega)$$

$$\Omega = \{ (\xi_1, \dots, \xi_N) : \xi_i = u \text{ or } d, \ i = 1, \dots, N \}$$

$|\Omega| = 2^N$; $N = 100$, $2^N \approx 10^{30}$

Industry perspective: Monte Carlo simulation is a method of computing option prices by averaging over "many" paths. How many? Typically, 100,000 common, maybe 1,000,000 or 10,000 for estimates what do we do?

(b) $1-q = \dfrac{(1+r)-d}{u-d} = \dfrac{5/4 - 1/2}{2 - 1/2} = 1/2, \quad q = 1/2$



$V(T,\omega) = (K - S(T,\omega))^+$

$V(n,\omega) = (5 - S(n,\omega))^+$

<u>n=3</u> observe: $V(3,\omega) = \begin{cases} 0, & \omega = uuu \\ 0, & \omega = uud, udu, duu \\ 3, & \omega = udd, dud, ddu \\ 9/2, & \omega = ddd \end{cases}$

$|\Omega| = 2^3 = 8$ but # nodes in $S(n,\omega)$ tree @ $n=3$ is # nodes $= n+1 = 4$

$V(3,\omega) = (K - S(3,\omega))^+ =: v(3, S(3,\omega))$

<u>ie</u> $V(3,\omega)$ only depends on ω through value of $S(3,\omega)$ and not directly on ω.

<u>n=2:</u> Compute $V(2,\omega)$

$S(2,\omega) = \begin{cases} 16, & \omega|_2 = uu \\ 4, & \omega|_2 = ud \text{ or } du \\ 1, & \omega|_2 = dd \end{cases}$

$$V(2,\omega) = \frac{1}{1+r}\left\{ q\, V(3;\, \underbrace{\omega|_2}_{\text{4 possible values}}, d) + (1-q) V(3;\, \omega|_2, u)\right\}$$

$$= \frac{1}{1+r}\left\{ q\, v(3, S(3;\omega|_2, d)) + (1-q)\, v(3, S(3;\omega|_2, u))\right\}$$

$$= \frac{1}{1+r}\left\{ q\, v(3, d\, \underbrace{S(2,\omega)}_{\text{3 possible values}}) + (1-q)\, v(3, u S(2,\omega))\right\}$$

$$=: v(2, S(2,\omega))$$

So $V(2,\omega)$ only depends on ω through $S(2,\omega)$.

$\underline{\omega|_2 = uu}$ $\Rightarrow$ $S(2,\omega) = 16$ and

$$V(2,\omega) = \frac{1}{1+1/4}\left\{ \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 \right\} = 0$$

$\omega|_2 = ud$ or du $\Rightarrow$ $V(2,\omega) = \frac{4}{5}\left\{ \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 3 \right\} = \frac{6}{5}$

$\omega|_2 = dd$ $\Rightarrow$ $V(2,\omega) = \frac{4}{5}\left\{ \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot \frac{9}{2} \right\} = 3$

$\underline{n=1:}$ $\omega|_1 = u$ or d

$$V(1,\omega) = \frac{1}{1+r}\left\{ q\, v(2, d S(1,\omega)) + (1-q)\, v(2, u S(1,\omega)) \right\}$$

$$=: v(1, S(1,\omega))$$

$\omega|_1 = u \Rightarrow V(1,\omega) = \frac{4}{5}\left\{ \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot \frac{6}{5} \right\} = 12/25$

$\omega|_1 = d \Rightarrow V(1,\omega) = \frac{4}{5}\left\{ \frac{1}{2}\cdot \frac{6}{5} + \frac{1}{2}\cdot 3 \right\} = 42/25$

$\underline{n=0:}$ $V(0) = \frac{1}{1+r}\left\{ q\, v(1, d S_0) + (1-q)\, v(1, u S_0) \right\}$

$$= v(0, S(0))$$

$$\Rightarrow V(0) = \frac{4}{5}\left\{\frac{1}{2}\cdot\frac{12}{25} + \frac{1}{2}\cdot\frac{42}{25}\right\} = \frac{108}{125}. \quad \square$$

Observation: In the example, we have
$$V(t_n, \omega) = v(t_n, S(t_n,\omega)),$$
for some function $v(t,x)$. Why?

Because $X(\omega) = v(t_N, S(t_N,\omega))$, $\forall \omega \in \Omega$, then
$$V(t_N,\omega) = v(t_N, S(t_N,\omega))$$

Risk-neutral pricing formula for $[t_{N-1}, t_N]$ gives

$$V(t_{N-1},\omega) = \frac{1}{1+r\delta t}\left\{(1-q)\, V(t_N; \omega|_{t_{N-1}}, u) + q\, V(t_N; \omega|_{t_{N-1}}, d)\right\}$$

$$= \frac{1}{1+r\delta t}\left\{(1-q)\, v(t_N, S(t_N; \omega|_{t_{N-1}}, u)) + q\, v(t_N, S(t_N; \omega|_{t_{N-1}}, d))\right\}$$

$$= \frac{1}{1+r\delta t}\left\{(1-q)\, v(t_N, u S(t_{N-1},\omega)) + q\, v(t_N, d S(t_{N-1},\omega))\right\}$$

$$=: v(t_{N-1}, S(t_{N-1},\omega))$$

By induction, $n = 0, 1, \ldots, N$:

$$\boxed{V(t_n, \omega) = v(t_n, S(t_n,\omega)) \text{ in binomial model}}$$

$\Rightarrow$ $V(t_n, \omega)$ only depends on path ω through the value $S(t_n, \omega)$.

What is special about $S(t_n, \omega)$?

$$S(t_n, \omega) = \begin{cases} u\, S(t_{n-1}, \omega), & \text{if } \xi_n = u \\ d\, S(t_{n-1}, \omega), & \text{if } \xi_n = d \end{cases}$$

where $\omega = (\omega|_{t_{n-1}}, \xi_n)$.

$\Rightarrow$ $S(t_n, \omega)$ only depends on $\omega|_{t_{n-1}}$ through $S(t_{n-1}, \omega)$ and not on all possible $(\# = 2^{n-1})$ $\omega|_{t_{n-1}}$.

$S(t_n, \omega)$ is an example of a discrete-time Markov process.

Review of Probability Theory

Some good books:
1. S. Ross
2. R. Durrett
3. Billingsley
4. Papoulis

Defn: Ω set, then a set of subsets of Ω, denoted $\mathcal{F}$, is called σ-algebra if

(i) $\emptyset \equiv \{\} \in \mathcal{F}$

(ii) $A \in \mathcal{F}$ then $A^c = \Omega \setminus A \in \mathcal{F}$

(iii) whenever $A_1, A_2, A_3, \ldots,$ in $\mathcal{F}$, then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

A probability measure on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ such that

(i) $\mathbb{P}(\Omega) = 1$

(ii) whenever $A_1, A_2, \dots$ in $\mathcal{F}$, $A_i \cap A_j = \phi$ for all $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Call $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space

Example $\Omega_N = \{ (\xi_1, \dots, \xi_N) : \xi_i = u \text{ or } d, \ i = 1, \dots, N \}$

Consider $\Omega_\infty = \{ (\xi_1, \xi_2, \xi_3, \dots) : \xi_i = u \text{ or } d, \ i \geq 1 \}$

Examples of σ-algebras:

(a) $\mathcal{F}_0 := \{ \phi, \Omega \}$. Is $\mathcal{F}_0$ a σ-algebra? yes. check defn.

Probability measures $\mathbb{P}$? $\mathbb{P}(\Omega) = 1$ and

$\mathbb{P}(\phi) = 0$. Why? $\mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \phi)$

$= \mathbb{P}(\Omega) + \mathbb{P}(\phi)$

$= 1 + \mathbb{P}(\phi)$.

(b) $\mathcal{F}_1 := \{ \phi, \Omega, A_u, A_d \}$. σ-algebra?

$A_u := \{ (u, \xi_2, \xi_3, \dots) : \xi_i = u \text{ or } d, \ i \geq 2 \}$

$A_d := \{ (d, \xi_2, \xi_3, \dots) : \xi_i = u \text{ or } d, \ i \geq 2 \}$

$A_d^c = A_u \in \mathcal{F}_1$

$A_u \cup A_d = \Omega \in \mathcal{F}_1$ $\therefore$ $\mathcal{F}_1$ is a σ-algebra.

(c) $\mathcal{F}_2 := \{ \phi, \Omega, A_{uu}, A_{ud}, A_{du}, A_{dd}, \underbrace{}_{(*)} \}$

$A_{dd} = \{ (d, d, \xi_3, \ldots) : \xi_i = u \text{ or } d, \; i \geq 3 \}$, etc.

$\mathcal{F}_2$ a σ-algebra?

$\overset{c}{A_{dd}} = A_{uu} \cup A_{ud} \cup A_{du}$

Not unless we add all complements and unions of $A_{dd}, A_{uu}, A_{du}, A_{ud}$.

One can show that $|\mathcal{F}_2| = 2^4$. (Exercise)

Probability measures on $(\Omega, \mathcal{F}_1)$:

$P(\phi) = 0$, $P(\Omega) = 1$, but $P(A_d) = q$, $0 \leq q \leq 1$
$\Rightarrow P(A_u) = 1 - q$.

$(\Omega, \mathcal{F}_2)$: Define P on $\mathcal{F}_2$ by choosing $P(A_{uu})$, $P(A_{ud})$, $P(A_{du})$, $P(A_{dd})$:

$P(A_{dd}) := q^2$, $P(A_{ud}) = P(A_{du}) := q(1-q)$,
$P(A_{uu}) := (1-q)^2$.

(d) $\mathcal{F}_3 := \{ \phi, \Omega, A_{uuu}, A_{uud}, A_{udu}, A_{duu}, A_{ddu}, A_{dud}, A_{udd}, A_{ddd}, * \}$

Can show $|\mathcal{F}_3| := 2^{(2^3)} = 256$

$(*)$ = all additional sets which need to be included to make $\mathcal{F}_3$ a σ-algebra.

(e) Get sequences of σ-algebras:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots \quad \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots$$

$\mathcal{F}_\infty :=$ set of all subsets of Ω described by finitely many ξ_i and then adding in all subsets of Ω needed to make $\mathcal{F}_\infty$ a σ-algebra

Eg. $A_{uu} \in \mathcal{F}_2$, $A_{uu} = \{(u,u,\xi_3,\ldots) : \xi_i = u, d \ \ i \geq 3\}$ $= A_{uuu} \cup A_{uud}$

$\in \mathcal{F}_3$ also.

<u>Example:</u> $\Omega = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

$\mathcal{B}[0,1] :=$ set of all subsets of $\Omega = [0,1]$ obtained by taking complements or unions of closed intervals in $[0,1]$, ie $[a,b]$, $0 \leq a \leq b \leq 1$.

<u>observation</u> $\mathcal{B}[0,1] \neq$ set of all subsets of Ω

$\mathcal{F}_\infty \neq$ set of all subsets of Ω

$\mathcal{B}[0,1]$ is called <u>Borel</u> <u>σ-algebra</u> for $[0,1]$

$\mathbb{P} : \mathcal{B}[0,1] \longrightarrow [0,1]$, $\mathbb{P}[a,b] = b - a$.

One can show $\mathbb{P}\{a\} = 0$

$\mathbb{P}(a,b) = b - a$

where $(a,b) := \{x \text{ in} \mathbb{R} : a < x < b\}$

Example: $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ is the (Borel) σ-algebra for $\mathbb{R}$ obtained by taking unions & complements of closed intervals in $\mathbb{R}$.

Can define on $\mathcal{F}$ by

$$P[a,b] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx, \quad -\infty < a \leq b < \infty$$

Probability measure associated with a normal distribution.

Random Variables & Probability Distributions

Defn: Given $(\Omega, \mathcal{F})$, a random variable is a function $X : \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega)$ such that X is Borel-measurable:

$$\underbrace{\{\omega \in \Omega : X(\omega) \in B\}}_{\substack{||| \\ X^{-1}(B) \text{ (shorthand)}}} \in \mathcal{F} \quad \text{for all } B \in \mathcal{B}(\mathbb{R})$$

For example, $B = [a,b] \in \mathcal{B}(\mathbb{R})$.

Example: Ω_∞ as above
$\{(\xi_1, \xi_2, \ldots) : \xi_i = u \text{ or } d, i \geq 1\}$

Consider binomial tree model stock price path, $S(n,\omega)$, $n = 0, 1, 2, \ldots$ $(\delta t = 1)$

$u = 2, \quad d = 1/2, \quad S_0 = 4$

<u>Random variables</u>:

$S(0,\omega) := 4 \quad \forall \quad \omega \in \Omega$

$$S(1,\omega) = \left\{ \begin{array}{ll} u\,S(0,\omega), & \xi_1 = u \\ d\,S(0,\omega), & \xi_1 = d \end{array} \right\} = \left\{ \begin{array}{ll} 8, & \xi_1 = u \\ 2, & \xi_1 = d \end{array} \right.$$

$$S(2,\omega) = \left\{ \begin{array}{ll} u\,S(1,\omega), & \xi_2 = u \\ d\,S(1,\omega), & \xi_2 = d \end{array} \right.$$

$$= \left\{ \begin{array}{ll} 16, & \omega|_2 = uu \\ 4, & \omega|_2 = ud, du \\ 2, & \omega|_2 = dd \end{array} \right.$$

Each $S(n,\omega)$ is a <u>random variable</u> on $(\Omega, \mathcal{F}_\infty)$ and also $(\Omega, \mathcal{F}_n)$

(check in later example)

<u>Defn</u>: Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. The <u>distribution measure on $\mathbb{R}$</u> defined X and $\mathbb{P}$ is

$$\mu_X(B) := \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\} = \mathbb{P}\{X^{-1}(B)\}$$

(X is measurable w.r.t. $\mathcal{F}$, so $X^{-1}(B) \in \mathcal{F}$ for all B in $\mathcal{B}(\mathbb{R})$).

<u>eg.</u> think of $B = [a,b)$ as typical element of $\mathcal{B}(\mathbb{R})$

<u>Example</u> (cont'd) Define $\mathbb{P}$ on $\mathcal{F}_n$ by

$\mathbb{P}(A) = (1-q)^k q^{n-k}$, $k = \#\ u$'s
$n-k = \#\ d$'s

$A = \{ (\underbrace{\xi_1, \ldots, \xi_n}_{\text{fixed}}, \xi_{n+1}, \ldots) : \xi_i = u \text{ or } d,\ i \geq n+1 \}$

Consider $n=2$ and $S(2,\omega)$:

$$X(\omega) \equiv S(2,\omega) = \begin{cases} 16, & \omega|_2 = uu \\ 4, & \omega|_2 = ud \text{ or } du \\ 2, & \omega|_2 = dd \end{cases}$$

$$B = \{16\} \Rightarrow X^{-1}(B) = \{\omega \in \Omega : X(\omega) = 16\}$$

$$= \{\omega \in \Omega : S(2,\omega) = 16\}$$

$$= \{\omega \in \Omega : \omega|_2 = uu\} = A_{uu} \in \mathcal{F}_2$$

$$\Rightarrow \mu_X(B) = \mathbb{P}\{X^{-1}(B)\} = \mathbb{P}(A_{uu}) = (1-q)^2 = \frac{1}{4}$$

where $\mathbb{P} : \mathcal{F}_2 \longrightarrow [0,1]$ was defined by

$\mathbb{P}(A_{uu}) = (1-q)^2$, $\mathbb{P}(A_{ud}) = \mathbb{P}(A_{du}) = (1-q)q$,
$\mathbb{P}(A_{dd}) = q^2$

$$B = \{4\} \Rightarrow X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$
$$= \{\omega \in \Omega : S(2,\omega) = 4\}$$
$$= A_{ud} \cup A_{du}$$

$$B = \{2\} = X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$
$$= \{\omega \in \Omega : S(2,\omega) = 2\}$$
$$= A_{dd}$$

$\Rightarrow \mu_X(\{4\}) = \mathbb{P}(\underbrace{A_{ud} \cup A_{du}}_{\text{disjoint},\ A_{ud} \cap A_{du} = \phi})$

$= \mathbb{P}(A_{ud}) + \mathbb{P}(A_{du}) = 2q(1-q) = \frac{1}{2}$

$\mu_X(\{2\}) = \mathbb{P}(A_{dd}) = q^2 = \frac{1}{4}$

$B = [3,5] \Rightarrow \mu_X[3,5] = \{\omega \in \Omega : X(\omega) \in [3,5]\}$
$= \{\omega \in \Omega : S(2,\omega) = 4\}$
$= 1/2.$

$\underline{X(\omega) = S(2,\omega)}$

$B = \{16\}$, $X^{-1}(B) = A_{uu} \in \mathcal{F}_2$ but not $\mathcal{F}_1$

$B = \{4\}$, $X^{-1}(B) = A_{ud} \cup A_{du} \in \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$ but not $\mathcal{F}_1$

$B = \{2\}$, $X^{-1}(B) = A_{dd} \in \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$, but not $\mathcal{F}_1$

$\Rightarrow$ $S(2,\omega)$ is measurable w.r.t. $\mathcal{F}_2$ (or $\mathcal{F}_3$ or $\mathcal{F}_4$ or ...) but not $\mathcal{F}_1$.

i.e. $S(2,\omega)$ is determined by $\mathcal{F}_2, \mathcal{F}_3, \ldots$ but not $\mathcal{F}_1$.

Probability Distribution & Density Functions

Given $(\Omega, \mathcal{F}, \mathbb{P})$ and random variable X on $(\Omega, \mathcal{F})$, then

$F_X : \mathbb{R} \to [0,1]$, $F_X(x) := \mathbb{P}\{X \leq x\}$

"$\{X \leq x\}$" $= \{\omega \in \Omega ; X(\omega) \leq x\}$

F_X is the cumulative probability

distribution function.

Given $F_X(x)$, know $\mu_X(B)$:

$$\mu_X(a,b] = \mathbb{P}\{a < X \leq b\} = \mathbb{P}\{X \leq b\} - \mathbb{P}\{X \leq a\}$$

$$\mu_X[a,b) = \lim_{n \to \infty} \mu_X(a - \tfrac{1}{n}, b]$$

<u>Defn</u>: Call $f_X : \mathbb{R} \to \mathbb{R}$ a <u>probability density function</u> for X if

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy \quad \underline{\text{ie}} \quad \frac{d}{dx} F_X(x) = f_X(x)$$

<u>Example</u>: $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\Phi(x) := \int_{-\infty}^{x} \phi(y)\,dy$

standard normal density + distribution functions.

<u>Expected Value of a Random Variable</u>

X is a random variable on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$.

- Ω <u>countable</u> (includes finite):

$$\mathbb{E}[X] := \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\omega)$$

- $\underline{\Omega}$ <u>uncountable</u> (eg, $\mathbb{R}$, $\mathbb{R}^d$, ...)

$$\mathbb{E}[X] := \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega)$$

provided we can explain what we mean by this "integral".

- Riemann integral (review)

$$\int_a^b f(x)\,dx$$

- Lebesgue integral (Shreve)

$$\int_\Omega X(\omega)\,d\mathbb{P}(\omega)$$, generalization of Riemann's idea.