16:642:621 MATH FINANCE I 10/2/2007

Note Title 10/2/2007

INFORMATION FLOW AND σ-ALGEBRAS

Example Binomial model & information flow

$T = 3$ (3 periods)

$$\Omega = \{ \omega = (\xi_1 \xi_2 \xi_3) : \xi_i = u \text{ or } d, \ 1 \le i \le 3 \}$$

Ω is market state space, ω is one "path"

t=0: $\omega \in \Omega$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$
(simplest possible σ-algebra for this Ω)

t=1: We know $\omega \in A_u$ or A_d

$$A_u = \{\underline{u}uu, \underline{u}ud, \underline{u}du, \underline{u}dd\}$$

$$= \{ (u \xi_2 \xi_3) : \xi_i = u \text{ or } d, \ i = 2, 3 \}$$

$$A_d = \{\underline{d}uu, \underline{d}ud, \underline{d}du, \underline{d}dd\}$$

$$\mathcal{F}_1 := \{\emptyset, \Omega, A_u, A_d\} \supset \mathcal{F}_0$$

t=2: Told outcome of market moves up to $t=2$. Then

$A_{uu} = \{\underline{uu}d, \underline{uu}u\}$ $\qquad$ $A_{du} = \{\underline{du}u, \underline{du}d\}$

$A_{ud} = \{\underline{ud}u, \underline{ud}d\}$ $\qquad$ $A_{dd} = \{\underline{dd}u, ddd\}$

Include all unions + complements to make a σ-algebra:

$$\mathcal{F}_2 = \{ \phi, \Omega, A_{uu}, A_{du}, A_{ud}, A_{dd}, A_{uu}^c, A_{du}^c, A_{ud}^c, A_{dd}^c,$$
$$A_{ud} \cup A_{du}, \quad A_{uu} \cup A_{dd},$$
$$A_{ud} \cup A_{dd}, \quad A_{du} \cup A_{uu}$$
$$A_u, A_d \}$$

$A_u = A_{uu} \cup A_{ud}$, $\quad A_d = A_{du} \cup A_{dd}$

$A_{uu}^c = A_{ud} \cup A_{du} \cup A_{dd}$, etc.

$|\mathcal{F}_2| = 2^{2^2} = 16$

<u>$t=3$:</u> $\quad \mathcal{F}_3 = \{$ all subsets of $\Omega \}$

$|\mathcal{F}_3| = 2^{2^3} = 256$ elements

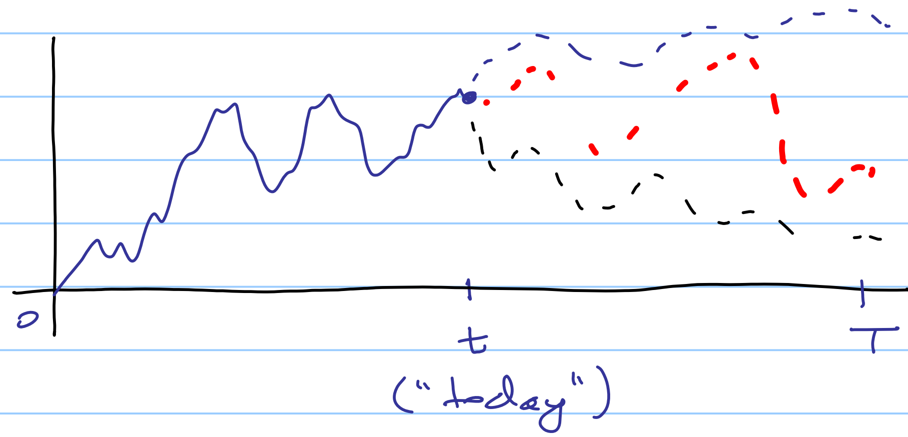
<u>obtain:</u> $\quad \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \quad (\subset \cdots, \text{if } T \geq 3)$

Sequence $\{\mathcal{F}(n)\}_{n=0,1,\cdots}$ is called a <u>filtration</u> of $\mathcal{F}$.

<u>Defn</u> Given Ω (non-empty set), let $T > 0$, spse $\forall\ t \in [0,T]$, you have a σ-algebra $\mathcal{F}(t)$ such that $\mathcal{F}(s) \subset \mathcal{F}(t)$ $\forall\ s \leq t$, then $\{\mathcal{F}(t)\}_{t \in [0,T]}$ is called a filtration.

Example $\Omega = C_0[0,T]$, the set of all cont's functions, $\omega : [0,T] \to \mathbb{R}$ such that $\omega(0) = 0$.

Spse $\bar{\omega} \in \Omega$. Can observe $\bar{\omega}(s)$, $0 \le s \le t$ but we do not know $\bar{\omega}(s)$, $t < s \le T$.



$$M := \{\omega \in \Omega : \max_{0 \le s \le t} \omega(s) \le 1\} \quad \text{"known @ } t\text{"}$$

$$P := \{\omega \in \Omega : \omega(T) > 0\} \quad \text{"unknown @ } t\text{"}$$

Can decide @ t whether $\bar{\omega} \in M$ (or not) but we cannot decide @ t whether $\bar{\omega} \in P$.

Information Learned by Observing a Random Variable

Defn: Spse X is a random variable. Then $\sigma(X)$, the sigma algebra generated by X is

$$\sigma(X) := \{\{X \in B\} : B \in \mathcal{B}(\mathbb{R})\}$$

where $\{X \in B\} := \{\omega \in \Omega : X(\omega) \in B\}$

$\mathcal{B}(\mathbb{R})$ = Borel σ-algebra generated by unions, complements of closed intervals, $[a, b] \subset \mathbb{R}$.

Example 3-period binomial model.

$u = 2, \; d = 1/2, \; S_0 = 4$

$$S(2, \omega) = \begin{cases} 16, & \omega = uud, uuu \\ 4, & \omega = udu, udd, duu, dud \\ 1, & \omega = ddu, ddd \end{cases}$$

$B = \{16\} \in \mathcal{B}(\mathbb{R})$; $S(2)$ is a random variable on Ω

$$\Rightarrow \{S(2) \in B\} = \{\omega \in \Omega : S(2, \omega) = 16\}$$

$$= \{uud, uuu\} = A_{uu}$$

$B = \{4\} \in \mathcal{B}(\mathbb{R})$

$$\Rightarrow \{S(2) \in B\} = \{\omega \in \Omega : S(2, \omega) = 4\}$$

$$= \{udu, udd, duu, dud\}$$

$$= A_{ud} \cup A_{du}$$

$B = \{1\} \in \mathcal{B}(\mathbb{R})$

$$\Rightarrow \{S(2)\} = \{\omega \in \Omega : S(2, \omega) = 1\}$$

$$= \{ddd, ddu\} = A_{dd}$$

$B = \phi \Rightarrow \{S(2) \in B\} = \phi$

$B = \mathbb{R} \Rightarrow \{S(2) \in B\} = \Omega$

$B = [4, 16] \Rightarrow \{S(2) \in B\} = A_{uu} \cup A_{ud} \cup A_{du}$

$\sigma(S_2) = \{\phi, \Omega, A_{uu}, A_{ud} \cup A_{du}, A_{dd}, +$ all unions and complements$\}$

$\subsetneq \mathcal{F}(2)$

<u>Reason:</u> "$\sigma(S_2)$ contains less information than $\mathcal{F}(2)$"

<u>Eg.</u> If we know $S(2, \omega) = 4$, then $\omega \in A_{ud} \cup A_{du}$ but we do not know whether $\omega \in A_{ud}$ or $\omega \in A_{du}$

<u>ie</u> $A_{ud} \cup A_{du} \in \sigma(S_2)$ but

$A_{ud} \notin \sigma(S_2)$, $A_{du} \notin \sigma(S_2)$

<u>measurability?</u> • $S(2, \omega)$ is <u>not</u> $\mathcal{F}(1)$ measurable.

Recall: $\mathcal{F}(1) = \{\phi, \Omega, A_u, A_d\}$. $S(2, \omega)$ <u>un</u>known any $\omega \in \mathcal{F}(1)$

• $S(2, \omega)$ is $\sigma(S_2)$ - measurable

$A_{ud} \cup A_{du} \in \underline{\sigma(S_2)}$, $\omega \in A_{ud} \cup A_{du}$

$\Rightarrow S(2, \omega)$ known $(= 4)$.

• $S(2,\omega)$ is $\mathcal{F}(2)$-measurable

$\omega \in A_{du}$ or $\omega \in A_{ud} \Rightarrow S(2,\omega)$ known $(=4)$.

Adapted Stochastic Processes & Portfolio Processes

Defn $(\Omega, \mathcal{F})$ given. A collection of random variables, $\{X(t)\}_{t \in [0,T]}$, is a stochastic process. (Implicit in definition is fact that each $X(t)$ is $\mathcal{F}$-measurable function, $X(t): \Omega \to \mathbb{R}$.) Call $X(t)$ an adapted process if $X(t)$ is $\mathcal{F}(t)$-measurable w.r.t. a filtration $\{\mathcal{F}(t)\}_{t \in [0,T]}$.

Example: $S(n)$ is binomial tree model stock price process and $\mathcal{F}(m)$, $m = 0, 1, 2, \ldots$ is filtration defined for this model, then $S(n)$ is $\mathcal{F}(n)$-measurable & so is adapted.

Example: $$V(t) = \Delta(t)\, S(t) + (\underbrace{V(t) - \Delta(t) S(t)}_{\text{cash}})$$

$S(t)$

Replicating portfolio value matching an option price (payoff $X(\omega)$ @ $t = T$.)

shares of stock $S(t)$

$\Delta(t)$ should be an adapted process (to $\mathcal{F}(t)$).

Independent Random Variables

Defn: $(\Omega, \mathcal{F}, \mathbb{P})$. Call sets $A, B \in \mathcal{F}$ **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Example: Binomial model, $T = 2$

$A = A_{uu} \cup A_{ud}$ (= "up @ t=1")
$B = A_{uu} \cup A_{du}$ (= "up @ t=2")

$\mathbb{P}\{u\} = 1-q$, $\mathbb{P}\{d\} = q$, $0 < q < 1$

$\mathbb{P}\{uu\} = (1-q)^2$, $\mathbb{P}(ud) = \mathbb{P}\{du\} = (1-q)q$
$\mathbb{P}\{dd\} = q^2$

Defines $\mathbb{P} : \mathcal{F} \to [0,1]$, as usual.

Are A, B independent or not?

Soln: $\mathbb{P}(A) = \mathbb{P}\{uu, ud\} = \mathbb{P}\{uu\} + \mathbb{P}\{ud\}$
$= (1-q)^2 + q(1-q) = (1-q)\{(1-q)+q\} = 1-q$
$\mathbb{P}(B) = \mathbb{P}\{uu, du\} = 1-q$

$\mathbb{P}(A) \cdot \mathbb{P}(B) = (1-q)^2$

$A \cap B = \{uu\}$, $\mathbb{P}(A \cap B) = (1-q)^2 = \mathbb{P}(A)\mathbb{P}(B)$ ✓.

Defn: $(\Omega, \mathcal{F}, \mathbb{P})$ probability space.

Given $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ σ-algebras.

Call $\mathcal{G}, \mathcal{H}$ independent σ-algebras

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \quad \forall \; A \in \mathcal{G}, \; B \in \mathcal{H}$$

ie A, B independent sets $\forall A \in \mathcal{G}, B \in \mathcal{H}$

Defn: If X, Y random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, call X, Y independent random variables if $\sigma(X), \sigma(Y)$ independent σ-algebras

Example: Binomial model. Are $S(1), S(2)$ independent? $T=2$

Soln: Heuristically, expect no: $S(1) = 8$ $\Rightarrow S(2) = 16$ or 4 but not $S(2) = 1$.

$$\mathbb{P}\{S(1) = 8 \text{ and } S(2) = 16\} \overset{?}{=} \mathbb{P}\{S(1) = 8\}\mathbb{P}\{S(2) = 16\}?$$

Observe: X, Y are independent random variables $\Leftrightarrow$

$$\mathbb{P}\{X \in B \text{ and } Y \in C\} = \mathbb{P}\{X \in B\} \cdot \mathbb{P}\{Y \in C\}$$

$\forall \; B, C \in \mathcal{B}(\mathbb{R})$ (essentially, closed intervals)

$\{S(1) = 8\} = \{uu, ud\}$, $\{S(1) = 8 \text{ and } S(2) = 16\}$
$\{S(2) = 16\} = \{uu\}$ $\qquad = \{uu\}$

$$\Rightarrow \mathbb{P}\{S(1) = 8\} = 1-q$$
$$\mathbb{P}\{S(2) = 16\} = (1-q)^2$$

$$\mathbb{P}\{S(1) = 8 \text{ and } S(2) = 16\} = (1-q)^2$$
$$\neq \mathbb{P}\{S(1) = 8\} \cdot \mathbb{P}\{S(2) = 16\} = (1-q)^3. \quad (0<q<1).$$

Pairs of random variables

Defn: X, Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$
then $(X,Y): \Omega \to \mathbb{R}^2$ is a (vector-valued) random variable. It has a joint distribution measure

$$\mu_{X,Y}(C) := \mathbb{P}\{(X,Y) \in C\} \quad \forall \quad C \in \mathcal{B}(\mathbb{R}^2)$$

where $\mathcal{B}(\mathbb{R}^2)$ is σ-algebra of subsets of $\mathbb{R}^2$ generated by all closed rectangles $[a_1, a_2] \times [b_1, b_2]$ & their unions & complements.

$\mu_{X,Y}: \mathcal{B}(\mathbb{R}^2) \to [0,1]$ is a probability measure.

The joint cumulative distribution function is

$$F_{X,Y}(a,b) := \mathbb{P}\{X \le a, Y \le b\}, \quad a, b \text{ in } \mathbb{R}$$

Call $f_{X,Y} \geq 0$ a density function for $\mu_{X,Y}$ if

$$\mu_{X,Y}(C) = \iint_C f_{X,Y}(x,y)\,dx\,dy$$

Equivalently:

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$$

i.e. $\dfrac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = f_{X,Y}(x,y)$.

<u>Marginal Probability Distribution</u>

Given $\mu_{X,Y}$, $F_{X,Y}$, $f_{X,Y}$, how do we determine $\mu_X, \mu_Y, F_X, F_Y, f_X, f_Y$?

$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy$; similarly $f_Y(y)$

$$F_X(x) = \mathbb{P}\{X \le x\} = \mathbb{P}\{X \le x, \underbrace{Y \in \mathbb{R}}_{Y < \infty}\}$$

$$= F_{X,Y}(x, \infty)$$

$$\mu_X(A) = \mathbb{P}\{X \in A\} = \mathbb{P}\{X \in A, Y \in \mathbb{R}\}$$

$$= \mu_{X,Y}(A \times \mathbb{R})$$

<u>Thm</u> X, Y random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. The following are equivalent:

(i) X, Y independent

(ii) $\mu_{X,Y}(A \times B) = \mu_X(A)\mu_Y(B)$ $\forall A, B \in \mathcal{B}(\mathbb{R})$

(iii) $F_{X,Y}(a,b) = F_X(a) F_Y(b)$, $\forall a, b \in \mathbb{R}$

(iv) $f_{X,Y}(a,b) = f_X(a) f_Y(b)$, $\forall a, b \in \mathbb{R}$

(if densities)

(v) $\mathbb{E}[e^{(uX + vY)}] = \mathbb{E}[e^{uX}]\,\mathbb{E}[e^{vY}]$
$\forall u, v \in \mathbb{R}$

$\Rightarrow$ (vi) $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Conditional Expectation

Example Binomial tree model, $T = N$.
$(\Omega, \mathbb{Q}, \mathcal{F})$ usual one;

$$\Omega = \{(\xi_1, \dots, \xi_N) : \xi_i = u \text{ or } d,\ 1 \le i \le N\}$$

$$\mathcal{F} = \mathcal{P}(\Omega) = \{\text{Set of all subsets of } \Omega\}$$

$$\mathcal{F}_n := \{(\xi_1, \dots, \xi_n) : \xi_i = u \text{ or } d,\ 1 \le i \le n\}$$

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_N = \mathcal{F}$$

$\omega = (\xi_1, \dots, \xi_N)$, then $\mathbb{Q}\{\omega\} = (1-q)^k q^{N-k}$,
$k = \#\ \xi_i = u$.

Conditional expectation of a random variable $X : \Omega \to \mathbb{R}$, given $\mathcal{F}(n)$:

$$\mathbb{E}[X \mid \mathcal{F}(n)](\omega) := \sum_{\omega|_n^c} (1-q)^{\#u(\omega|_n^c)} \cdot q^{\#d(\omega|_n^c)} X(\omega)$$

where:

$$\omega = (\underbrace{\xi_1, \dots, \xi_n}_{\omega|_n}, \underbrace{\xi_{n+1}, \dots, \xi_N}_{\omega|_n^c})$$

$\#u(\omega|_n^c) = \#$ up moves in $(\xi_{n+1}, \dots, \xi_N)$
$\#d(\omega|_n^c) = \#$ down " " "

$$\mathbb{E}[X \mid \mathcal{F}(n)](\omega) = \mathbb{E}[X \mid \mathcal{F}(n)](\underline{\omega|_n}, \omega|_n^c)$$

$$= \mathbb{E}[X \mid \mathcal{F}(n)](\omega|_n),$$

$$\omega|_n \in \mathcal{F}(n).$$

This is an $\mathcal{F}(n)$-measurable random variable.

<u>$n=0$:</u> $\mathbb{E}[X \mid \mathcal{F}(n)] = \mathbb{E}[X \mid \mathcal{F}(0)]$

$$= \sum_{\omega \in \Omega} \mathbb{Q}(\omega) X(\omega) = \mathbb{E}[X]$$

<u>$n=N$:</u> $\mathbb{E}[X \mid \mathcal{F}(N)] = X$, an $\mathcal{F}$-measurable random variable.

<u>Defn</u> Given $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \subset \mathcal{F}$, and random variable X, then

$\mathbb{E}[X \mid \mathcal{G}]$ is the <u>conditional expectation</u> of X given $\mathcal{G}$ if

(i) $\mathbb{E}[X \mid \mathcal{G}]$ is $\mathcal{G}$-measurable

(ii) $\displaystyle\int_A \mathbb{E}[X \mid \mathcal{G}](\omega)\, d\mathbb{P}(\omega) = \int_A X(\omega)\, d\mathbb{P}(\omega)$

(Partial averaging property)

When $\mathcal{G} = \sigma(Y)$, some random variable Y, then

$$\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X \mid Y].$$

<u>Example</u> Binomial model, $T=2$, $\mathcal{F} = \mathcal{F}(2)$,

$\mathcal{G} = \sigma(S(1))$

Show ~~that~~ $\mathbb{E}[S(2) \mid S(1)]$ is an $\mathcal{F}(1)$ measurable random variable.

<u>Soln:</u> $\sigma(S(1)) = \{\phi, \Omega, A_u, A_d\} = \mathcal{F}(1)$

$\mathbb{E}[S(2) \mid S(1)] = \mathbb{E}[S(2) \mid \mathcal{F}(1)]$

formula

$= (1-q) S(2; \omega|_1, u) + q S(2; \omega|_1, d)$

$\mathbb{E}[S(2) \mid S(1)](\omega)$ is determined by $\omega|_1$

<u>ie</u> is $\mathcal{F}(1)$- measurable random variable.

<u>Properties</u> of <u>Conditional Expectation</u>

<u>Theorem</u> $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subset \mathcal{F}$

(i) $\mathbb{E}[\alpha X + \beta Y \mid \mathcal{G}] = \alpha \mathbb{E}[X \mid \mathcal{G}] + \beta \mathbb{E}[Y \mid \mathcal{G}]$

"Linearity"

(ii) $\mathbb{E}[XY \mid \mathcal{G}] = X \mathbb{E}[Y \mid \mathcal{G}]$ if X $\mathcal{G}$-measurable

("Taking out what is known")

(iii) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}] = \mathbb{E}[X \mid \mathcal{H}]$ if $\mathcal{H} \subset \mathcal{G}$

("Tower property")

(iv) $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$ if X independent of $\mathcal{G}$

(v) Jensen's inequality:

$$\mathbb{E}[\varphi(X) \mid \mathcal{G}] \geq \varphi(\mathbb{E}[X \mid \mathcal{G}])$$

where φ is a convex function:

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$$

$\forall\ \lambda \in [0,1]$, $\quad x_1 < x_2$