Mathematical Description of a Basic Financial Model

Definition: A market (or economy) has p tradable assets, labelled $1, \ldots, p$.

(1) Trading times: $0 = t_0 < t_1 < \cdots < t_n = T$

$\mathcal{T} = \{t_0, t_1, \ldots, t_n\}$

n-period model (discrete-time)

$t = 0$ is "today"

T is terminal date or maturity

Note: Use $\mathcal{T} = [0, T]$ for continuous-time models in Shreve II.

(2) Sample (state) Space: $\Omega = \{\omega_1, \ldots, \omega_m\}$

Finite-state model ($m < \infty$)

Note: Use $\Omega = \mathbb{R}^d$, etc, for continuous-state models in Shreve II.

Each $\omega \in \Omega$ representing a "state" of the market (or economy).

(3) Price Functions (processes):

$\{S_1(t, \omega), \ldots, S_p(t, \omega)\}$ $\quad t \in \mathcal{T}, \omega \in \Omega$

$S_i(t,\omega)$ in $\mathbb{R}$

At time $t=0$, $S_i(0,\omega) = S_i(0)$, $i=1,\dots,p$.
Prices independent of $\omega\in\Omega$ at $t=0$.

(4) Also assume that the market contains a "bank account" or "money market security":

$B(t,\omega) > 0$ for all (t,ω)
$B(0,\omega) = B_0$

Distinguished asset.

Usually assume $B(t,\omega) = B(t)$, for all $\omega\in\Omega$

Examples: $\Delta t = t_{k+1} - t_k$, often choose

$$B(t+\Delta t) = \begin{cases} (1+r\Delta t)B(t), & \text{or} \\ e^{r\Delta t} B(t) \end{cases}$$

Simple compounding or continuous.

r is the risk-free interest rate, $r>0$.

Assume $B(t,\omega)$ is one of the $S_i(t,\omega)$ (usually $i=1$).

Portfolio processes (Trading Strategies)

Definition: A trading strategy lists # shares an investor holds in each asset, $1,\dots,p$, at each time $t\in\mathcal{T}$ and market state $\omega\in\Omega$:

$\phi_1(t,\omega), \ldots, \phi_p(t,\omega)$ in $\mathbb{R}$, (t,ω) in $\mathcal{T} \times \Omega$

$$\Phi(t,\omega) = \begin{pmatrix} \phi_1(t,\omega) \\ \vdots \\ \phi_p(t,\omega) \end{pmatrix} \in \mathbb{R}^p$$

A portfolio value process is the total value of portfolio at each $(t,\omega) \in \mathcal{T} \times \Omega$:

$$\Pi(t,\omega) = \sum_{i=1}^{p} \phi_i(t,\omega) S_i(t,\omega)$$

$$= (\phi_1(t,\omega), \ldots, \phi_p(t,\omega)) \begin{pmatrix} S_1(t,\omega) \\ \vdots \\ S_p(t,\omega) \end{pmatrix}$$

$$= \Phi(t,\omega) \cdot A(t,\omega)$$

Note that $\phi_i(0,\omega) = \phi_i(0)$, independent of $\omega \in \Omega$

Trading strategy is known at $t=0$.

One also assumes $\Phi(t_1,\omega) = \Phi(0)$

i.e. $\phi_i(t_1,\omega) = \phi_i(0)$, $i = 1, \ldots, p$

$\phi_i(t,\omega)$ = # shares of i^{th} security i.

$\phi_i < 0$: investor is short asset

$\phi_i > 0$: " " long asset

<u>Arbitrage:</u> We'll specialize to $n=1$ (one period model): $t_0 = 0$, $t_1 = T$

(Often texts assume $\Delta t = 1$, $T = 1$.)

<u>Defn:</u> An <u>arbitrage</u> is a trading strategy

$\Phi^* = (\phi_1, \phi_2, \ldots, \phi_p)$ such that

(a) $\Phi \cdot A(0) \equiv \Pi(0) < 0$ <u>and</u> $\Phi \cdot A(T,\omega) \equiv \Pi(T,\omega)$ is $\geq 0 \;\forall\, \omega$

or (b) $\Phi \cdot A(0) \equiv \Pi(0) = 0$ and

$\Phi \cdot A(T,\omega) \equiv \Pi(T,\omega) \geq 0 \;\forall\, \omega$ in Ω and $\Pi(T,\omega') > 0$ for <u>some</u> ω' in Ω

<u>Example:</u> Suppose $B(T) = (1+rT)B_0$, $B(0) = B_0$
$S_2(t,\omega) = S(t,\omega)$.

Bank account, one stock. $\Omega = \{up, down\}$

$$S(0,\omega) = S_0, \qquad S(T,\omega) = \begin{cases} uS_0, & \omega = up \\ dS_0, & \omega = down \end{cases}$$

Assume $S_0 > 0$, $B_0 > 0$, $0 < d < u$, $\underset{\text{usually}}{(d<1,\ u>1)}$

<u>Claim:</u> Market is arbitrage-free $\Longleftrightarrow$

$$d < \underbrace{1+rT}_{=B(T)/B(0)} < u$$

<u>Proof:</u> We'll prove "$\Rightarrow$". ("$\Leftarrow$" is exercise.)

Suppose $1+rT \leq d < u$. Get an arbitrage by borrowing @ r and investing in S.

Suppose $d < u \leq 1+rT$. Get an arbitrage by selling S short and investing @ r.

<u>Formal Proof:</u> Suppose $1+rT \leq d < u$.

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -S_0 \\ B_0 \end{pmatrix}, \quad A(t,\omega) = \begin{pmatrix} B(t) \\ S(t,\omega) \end{pmatrix}$$

<u>$t=0$:</u> $\Pi(0) = \Phi \cdot A(0) = \begin{pmatrix} -S_0 \\ B_0 \end{pmatrix} \cdot \begin{pmatrix} B(0) \\ S(0) \end{pmatrix} = 0$

<u>$t=T$:</u> $\Pi(T,\omega) = \Phi \cdot A(T,\omega) = \begin{pmatrix} -S_0 \\ B_0 \end{pmatrix} \cdot \begin{pmatrix} B(T) \\ S(T,\omega) \end{pmatrix}$

$$= \begin{pmatrix} -S_0 \\ B_0 \end{pmatrix} \cdot \begin{pmatrix} B_0(1+rT) \\ S(T,\omega) \end{pmatrix}$$

<u>$\omega = u$:</u> $\Pi(T,u) = \begin{pmatrix} -S_0 \\ B_0 \end{pmatrix} \cdot \begin{pmatrix} B_0(1+rT) \\ uS_0 \end{pmatrix}$

$$= B_0 S_0 \left(-(1+rT) + u\right) > 0$$

by assumption

<u>$\omega = d$:</u> $\Pi(T,d) = \begin{pmatrix} -S_0 \\ B_0 \end{pmatrix} \cdot \begin{pmatrix} B_0(1+rT) \\ dS_0 \end{pmatrix}$

$$= B_0 S_0 \left(-(1+rT) + d\right) \geq 0$$

$\Rightarrow$ type (b) arbitrage.

Similarly get arbitrage if $d < u \leq 1+rT$. $\square$

<u>Continuous Compounding:</u> No arbitrage $\Leftrightarrow$ $d < e^{rT} < u$

where $e^{rT} = B(T)/B(0)$.

Some Simple Derivative Securities

Examples: (1) Forward Contract: Defined by "payoff"

$$FC(T,\omega) := S(T,\omega) - K \quad \forall\, \omega \text{ in } \Omega$$

Delivery price K, maturity T, underlying $S(t,\omega)$.

(2) $$C(T,\omega) := (S(T,\omega) - K)^+ = \max\{S(T,\omega) - K, 0\}$$

Payoff function for a call option (European style).

(3) $$P(T,\omega) := (K - S(T,\omega))^+$$

Payoff function for a put option (European style)

What are prices of these options at $t=0$?

(2), (3) "complicated". We'll see during semester

(1) We'll see (exercise): no-arbitrage pricing

$$FC(0) = S(0) - \frac{B(0)}{B(T)} K \quad \text{(no arbitrage proof)}$$

where $B(0)/B(T) = (1+rT)^{-1}$ or e^{-rT}.

Proof: Suppose $FC(0) < S(0) - \frac{B(0)}{B(T)}K$

Assume $B(0)=1$, $B(T) = 1+rT$. Then choose

$$\Phi = \begin{pmatrix} K(1+rT)^{-1} \\ -1 \\ 1 \end{pmatrix}, \quad A(t,\omega) = \begin{pmatrix} B(t) \\ S(t,\omega) \\ FC(t,\omega) \end{pmatrix}$$

$$\Pi(t,\omega) = \Phi \cdot A(t,\omega) = \begin{pmatrix} K/(1+rT) \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} B(t) \\ S(t,\omega) \\ FC(t,\omega) \end{pmatrix}$$
$$= (1+rT)^{-1}KB(t) - S(t,\omega) + FC(t,\omega)$$

$\Pi(0) = (1+rT)^{-1}K - S(0) + FC(0) < 0$ by assumption

$$\Pi(T,\omega) = (1+rT)^{-1}KB(T) - S(T,\omega) + FC(T,\omega)$$
$$= K - S(T,\omega) + FC(T,\omega) = 0 \text{ by defn.}$$

$\Rightarrow$ arbitrage.

Similarly, get an arbitrage if
$FC(0) > S(0) - \frac{B(0)}{B(T)}K$. $\square$

Defn: The forward price $F(0)$ of the asset S is the delivery price K such that $FC(0) = 0$

ie $\quad K = \frac{B(T)}{B(0)} S(0) =: F(0)$

No-arbitrage pricing and replicating portfolios

One-period model.

Suppose derivative security has payoff $V(T,\omega)$, $\omega \in \Omega$.

$V(0) = ?$

A portfolio $\Phi = (\phi_1, \ldots, \phi_p)^*$ is called a a **replicating portfolio** for the derivative security if

$$V(T,\omega) = \Phi \cdot A(T,\omega) \quad \text{for all } \omega \in \Omega.$$

Claim If the market $\{S_1, \ldots, S_p, V\}$ is arbitrage-free then

$$V(0) = \Phi \cdot A(0)$$

Proof: Same idea as for forward contract. ∎

Next Day! Binomial Branch model.