16:642:621 MATH FINANCE I 10/10/2007

Note Title 10/10/2007

INTRODUCTION TO BROWNIAN MOTION

- Physical phenomenon observed by R. Brown (1827)
- LOUIS BACHELIER, Ph.D. THESIS, 1900, "THEORIE DE LA SPECULATION"
  ARITHMETIC BROWNIAN MOTION — MODEL FOR STOCK MARKET PROCESS (INDEX)

  (TODAY, CAC 40)
- 1905 A. EINSTEIN, PHOTOELECTRIC EFFECT

SYMMETRIC RANDOM WALKS

$$\Omega = \{ \omega = (\xi_1, \xi_2, \dots) : \xi_i = +1 \text{ or } -1, \ i \geq 1 \}$$

$\mathcal{F}$ as before

$\{X_j\}_{j=1}^{\infty}$ sequence of random variables on $(\Omega, \mathcal{F})$, iid (independent, identically distributed)

$$X_j(\omega) = \begin{cases} 1, & \xi_j = 1 \\ -1, & \xi_j = -1 \end{cases}$$ (Bernoulli)

$\mathbb{P}\{X_j = 1\} = 1-p$, $\mathbb{P}\{X_j = -1\} = p$. Choose $p = 1/2$.

<u>Defn:</u> <u>Symmetric</u> <u>random</u> <u>walk</u>, $\{M_k\}_{k=0}^{\infty}$

$$M_0 := 0, \quad M_k := \sum_{j=1}^{k} X_j, \quad k \geq 1$$

<u>Exercise:</u>

(*) $\{M_k\}_{k=0}^{\infty}$ has <u>independent</u> <u>increments</u> <u>property</u>:

<u>Defn:</u> For any $0 = k_0 < k_1 < k_2 < \cdots < k_m < \infty$, a process $\{Y_k\}_{k=0}^{\infty}$ has independent increments if the following are independent random variables:

$$Y_{k_1} - Y_{k_0},\ Y_{k_2} - Y_{k_1},\ \dots,\ Y_{k_m} - Y_{k_{m-1}}$$

Observations for $\{M_k\}_{k=0}^{\infty}$:

- $\mathbb{E}[M_{k_{i+1}} - M_{k_i}] = 0$
- $\mathrm{Var}[M_{k_{i+1}} - M_{k_i}] = k_{i+1} - k_i$

(*) Exercise: check values of mean, variance for any $0<p<1$.

Soln: $\mathbb{E}[X_j] = (1)\,\mathbb{P}\{X_j = 1\} + (-1)\,\mathbb{P}\{X_j = -1\}$

$$= (1-p) + (-1)p$$

$$= 1 - 2p = 0, \text{ when } p = 1/2.$$

$$\mathrm{Var}[X_j] = \mathbb{E}\{(X_j - \mathbb{E}[X_j])^2\} = \mathbb{E}[X_j^2]$$

$$= (1-p) + p = 1$$

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

$$\Rightarrow \mathbb{E}[M_{k_{i+1}} - M_{k_i}] = 0$$

$$\begin{aligned}
\mathrm{Var}[M_{k_{i+1}} - M_{k_i}] &= \mathbb{E}[(M_{k_{i+1}} - M_{k_i})^2] \\
&= \mathbb{E}\left[\left(\sum_{j=k_i+1}^{k_{i+1}} X_j\right)^2\right] \\
&= \sum_{j=k_i+1}^{k_i} \mathbb{E}[X_j^2] \quad (X_j, X_k \text{ independent when } j \neq k) \\
&= \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i
\end{aligned}$$

$\square$

<u>Quadratic Variation</u> $[M,M]_k$, $k \geq 0$, stochastic process, defined by

$$[M,M]_k := \sum_{j=1}^{k} (M_j - M_{j-1})^2$$

For the random walk, $\{M_k\}_{k \geq 0}$, can show that

$$[M, M]_k = \sum_{j=1}^{k} X_j^2 = k.$$

Martingale Property of Random Walk

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be as above, with filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ defined, as usual,

$$\mathcal{F}_0 = \{\phi, \Omega\}$$

take all unions + complements

$\mathcal{F}_n :=$ σ-algebra generated by all subsets $A_{\xi_1 \xi_2 \cdots \xi_n} \subset \Omega$, where

$$A_{\xi_1 \xi_2 \cdots \xi_n} := \{\omega \in \Omega : \omega = (\xi_1, \xi_2 \cdots \xi_n \eta_{n+1} \eta_{n+2} \cdots)$$
$$\eta_j = \pm 1,\ j \geq n+1\}$$

("atom")

$\mathcal{F} = \mathcal{F}_\infty :=$ σ-algebra generated by

$$\bigcup_{n=0}^{\infty} \mathcal{F}_n .$$

Defn: A discrete-time process $X(n,\omega)$, $n \geq 0$, $\omega \in \Omega$, on $(\Omega, \mathcal{F})$ is called a martingale with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and $\{\mathcal{F}(n)\}_{n=0}^{\infty}$ filtration of $\mathcal{F}$ if

$$\mathbb{E}_{\mathbb{P}}[X(n) \mid \mathcal{F}(m)] = X(m), \quad n \geq m \geq 0.$$

Example: Symmetric random walk is a martingale.

Soln: $\mathbb{E}[M_n \mid \mathcal{F}_m] = \mathbb{E}[(M_n - M_m) + M_m \mid \mathcal{F}_m]$

$$= \mathbb{E}[\underbrace{M_n - M_m}_{\sum_{j=m+1}^{n} X_j} \mid \mathcal{F}_m] + \mathbb{E}[M_m \mid \mathcal{F}_m]$$

independent　　　　is $\mathcal{F}_m$-measurable

$$= \mathbb{E}[M_n - M_m] + M_m = 0 + M_m = M_m. \quad \square$$

MIDTERM 10/23; FINAL 12/18

Continuous-time, scaled symmetric random walk

Fix $n \geq 1$, integer.

$W^{(n)}(t) := \frac{1}{\sqrt{n}} M_{nt}$, when nt integer ≥ 0.

Spse $nt \neq$ integer. Then define $W^{(n)}(t)$ using linear interpolation:

Spse $s < t < u$, where s, t in $\mathbb{R}$ so that

ns = largest integer $< nt$
nu = smallest " $> nt$

$$\Rightarrow W^{(n)}(t) := \frac{u-t}{u-s} W^{(n)}(s) + \frac{t-s}{u-s} W^{(n)}(u)$$

Properties of Scaled symmetric random walk:

- Independent increments: $0 = t_0 < t_1 < \cdots < t_m$, nt_j in $\mathbb{Z}$, ~~then~~

$$W^{(n)}(t_1) - W^{(n)}(t_0), \ldots, W^{(n)}(t_m) - W^{(n)}(t_{m-1}) \text{ independent.}$$

- Mean + Variance of increments

$$\mathbb{E}\left[W^{(n)}(t_j) - W^{(n)}(t_{j-1})\right] = 0 , \quad j \geq 0$$

$$\mathrm{Var}\left[W^{(n)}(t_j) - W^{(n)}(t_{j-1})\right] = t_j - t_{j-1}$$

Limiting Distribution of the Scaled Random Walk

Central Limit Theorem Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$S_n := \sum_{j=1}^{n} X_j \,, \qquad \bar{S}_n := \frac{1}{\sigma\sqrt{n}}(S_n - n\mu), \text{ where}$$

$$\mu := \mathbb{E}[X_j] \,, \quad \sigma^2 = \mathrm{Var}[X_j] \,, \quad j \geq 1 .$$

Define $F_n(x) := \mathbb{P}\{\bar{S}_n \leq x\}$, x in $\mathbb{R}$, $n \geq 1$.

Then $\lim_{n\to\infty} F_n = \Phi_{0,1}$, where

$$\Phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} \, dx$$

(*) <u>Exercise</u> : Justify the Central Limit Theorem.

<u>Application to Scaled Random Walk</u>

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \frac{1}{\sqrt{n}} \sum_{j=1}^{nt} X_j \,, \qquad nt \text{ in } \mathbb{Z} \quad (t > 0)$$

$\{X_j\}_{j=1}^{\infty}$ is a sequence of Bernoulli random variables with

$$\mathbb{P}\{X_j = 1\} = 1-p, \quad p = 1/2.$$
$$\mathbb{P}\{X_j = -1\} = p$$

For theorem, $\mu = 0$, variance $\sigma^2 = 1$.

$$W^{(n)}(t) = \frac{1}{\sqrt{nt}} \sum_{j=1}^{nt} \sqrt{t}\, X_j = \sqrt{t}\, \frac{(S_{nt} - nt \cdot 0)}{\sqrt{nt}}$$

where "0" $= \mathbb{E}[\sqrt{t} X_j] =$ "μ"

"$\sqrt{t}$" $= \sqrt{Var[\sqrt{t} X_j]} =$ "σ"

$$\Rightarrow \lim_{n\to\infty} \text{Distribution}\left(\frac{W^{(n)}(t)}{\sqrt{t}}\right) = \Phi_{0,1}$$

$$\lim_{n\to\infty} \text{Distribution}(W^{(n)}(t)) = \Phi_{0,t}$$

Conclusion: $\boxed{\text{As } n \to \infty,\ W^{(n)}(t) \text{ becomes a normal random variable with mean } 0,\text{ variance } t}$

Introduction to Brownian Motion

$$\boxed{W(t) = \lim_{n \to \infty} W^{(n)}(t)}$$

Can view Brownian Motion as the stochastic process obtained by taking limit of scaled random walks, as $n \to \infty$.

Defn: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose, for each $\omega \in \Omega$, there is a continuous function $W(t, \omega) = W(t)$ so that $W(0) = 0$. Then $W(t)$ is called

Brownian motion if

- $W(t)$ has independent increments:

  For any $0 = t_0 < t_1 < t_2 < \cdots < t_m$, then

  $W(t_1) - W(t_0), \ldots, W(t_m) - W(t_{m-1})$ are independent,

- Increments of $W(t)$ are normally distributed:

$$E[W(t_j) - W(t_{j-1})] = 0, \quad j \geq 1$$

$$Var[W(t_j) - W(t_{j-1})] = t_j - t_{j-1}$$

Interpretations of Brownian Motion

- $\omega \in \Omega$ is a sequence of "infinitely fast" coin tosses.

- given $\omega \in \Omega$, have continuous sample path $W(t,\omega)$, $t \geq 0$.

- Identify $\omega \in \Omega$ with $W(t,\omega)$, where $W(\cdot,\omega)$ in $C[0,\infty)$ (the set of $\mathbb{R}$-valued continuous functions of t in $[0,\infty)$).

So we usually choose $\Omega = C[0,\infty)$.

Brownian Motion & Information Flow

$W(t)$ is Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$

Call a collection $\{\mathcal{F}(t)\}_{t \geq 0}$ of σ-algebras of subsets of Ω a filtration for $W(t)$ if

(i) $\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F}$, $0 \leq s < t$

(ii) $W(t)$ is $\mathcal{F}(t)$-measurable, ie $W(t)$ is an adapted process w.r.t. the

filtration

Recall: $W(t)$ is $\mathcal{F}(t)$-measurable if

$$\{W(t) \in B\} \in \mathcal{F}(t) \quad \forall \, B \in \mathcal{B}(\mathbb{R}).$$

$$\{\omega \in \Omega : W(t,\omega) \in B\}$$

(Borel sets, eg $[a,b]$.)

(iii) $W(u) - W(t)$ is independent of $\mathcal{F}(t)$ for $0 \leq t < u$.

We call a stochastic process $\Delta(t)$ adapted to $\mathcal{F}(t)$ if $\Delta(t)$ is $\mathcal{F}(t)$-measurable.

Call $\mathcal{F}(t)$ the filtration generated by $W(t)$ if $\mathcal{F}(t) = \sigma(W(t))$, the smallest σ-algebra such that $W(t)$ is $\mathcal{F}(t)$-measurable.

Martingale Property of Brownian Motion

Theorem Brownian motion is a martingale.

Soln: Given $0 \le s < t$, then

$$\mathbb{E}[W(t) \mid \mathcal{F}(s)] = \mathbb{E}[W(t) - W(s) \mid \mathcal{F}(s)] + \mathbb{E}[W(s) \mid \mathcal{F}(s)]$$

$$= 0 + W(s) = W(s). \quad \square$$

Quadratic Variation

Defn: Let $f : [0,T] \to \mathbb{R}$ be a function. Then its quadratic variation is

$$[f,f](T) := \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n} \left| f(t_{j+1}) - f(t_j) \right|^2$$

where $\Pi := \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$.

$$\|\Pi\| := \max_{0 \le j \le n-1} |t_{j+1} - t_j|$$

(*) Exercise: Suppose f is continuously differentiable on $(0,T)$, continuous on $[0,T]$, then

$$[f,f](T) = 0$$

Hint: Uses mean value theorem,

$$f(t_{j+1}) - f(t_j) = f'(t_j^*) \cdot (t_{j+1} - t_j)$$

for some $t_j^* \in (t_j, t_{j+1})$. $\square$

Thm Let $W(t)$ be Brownian motion. Then $[W,W](t)$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$[W,W](t) = t, \quad t \geq 0$$

Consequence: $W(t)$ sample paths are not differentiable.

Sometimes write: $dW(t) \cdot dW(t) = dt$

(See texts of Wilmott, Hull, ...)

<u>Quadratic Co-variation</u>

$$[f, g](t) := \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \big(f(t_{j+1}) - f(t_j)\big)\big(g(t_{j+1}) - g(t_j)\big)$$

Can show:

ⓐ $[W, \tau](t) = 0$, if $\underline{\tau(t) = t}$, $t \geq 0$.

ⓑ $[\tau, \tau](t) = 0$

Usually see written: $dW(t) \cdot dt = 0$, $dt \cdot dt = 0$.

Heuristic Justification of $[W,W](t) = t$

Define $Q_\Pi := \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j) \right)^2$, $\omega \in \Omega$

$$\mathbb{E}[Q_\Pi] = \sum_{j=0}^{n-1} \mathrm{Var}\left[W(t_{j+1}) - W(t_j) \right] = \sum_{j=0}^{n-1} t_{j+1} - t_j = t$$

$\Pi = \{t_0, t_1, \ldots, t_n\}$, $0 = t_0 < t_1 < \cdots < t_n = t$

$$\mathrm{Var}[Q_\Pi] \underset{\text{exercise}}{=} \mathbb{E}\left[(W(t_{j+1}) - W(t_j))^4 \right] - 2\,\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^2 \cdot (t_{j+1} - t_j) \right] + (t_{j+1} - t_j)^2$$

Exercise: $\mathbb{E}[Z^4] = 3\,\mathrm{Var}[Z]^2$, for normal random variable with mean zero.

$$\Rightarrow \mathrm{Var}[Q_\Pi] \overset{\text{exercise}}{=} \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2$$

$$\leq 2 \|\Pi\| \sum_{j=0}^{n-1} (t_{j+1} - t_j)$$

$$= 2 \|\Pi\| \cdot t \longrightarrow 0 \quad \text{as } \|\Pi\| \to 0.$$

$$\Rightarrow \lim_{\|\Pi\|} \mathrm{Var}[Q_\Pi] = 0$$

$$\Rightarrow \lim_{\|\Pi\| \to 0} Q_\Pi = \mathbb{E}[Q_\Pi] = t. \qquad \square$$

Markov Property of Brownian Motion

Defn: Given a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\mathcal{F}(t)$, $X(t)$ adapted to $\mathcal{F}(t)$. Then $X(t)$ is Markov if, whenever f is a function (Borel measurable on $\mathbb{R}$), there is a function g (" " ") such that

$$\mathbb{E}[f(X(t)) \mid \mathcal{F}(s)] = g(X(s)), \quad 0 \leq s \leq t.$$

Thm Suppose $W(t)$ is Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}(t)$ is a filtration for $W(t)$. Then $W(t)$ is a Markov process.