1. In each part below give the precise definition in one or more full sentences.
(a) The span of a set of vectors \( S = \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \);
(b) A linearly independent set of vectors \( S = \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \);
(c) A subspace of \( \mathbb{R}^n \);
(d) A basis of a subspace \( W \) of \( \mathbb{R}^n \);
(e) An eigenvector and corresponding eigenvalue of a square matrix \( A \);
(f) An eigenspace of a square matrix \( A \).

2. Suppose that \( A \) is an \( m \times n \) matrix.
(a) Define the null space \( \text{Null}(A) \) of \( A \).
(b) Show that \( \text{Null}(A) \) is a subspace of \( \mathbb{R}^n \) by checking the conditions in the definition of a subspace.
(c) Define the column space \( \text{Col}(A) \) of \( A \).
(d) Show that \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^m \) by checking the conditions in the definition of a subspace.

3. Find the \( A = LU \) factorization (that is, find \( L \) and \( U \)) of \( A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 2 & -2 & 3 \\ 0 & 3 & 2 & 7 \end{bmatrix} \). Then use it to solve \( Ax = \begin{bmatrix} 3 \\ -4 \\ 10 \\ 12 \end{bmatrix} \) by solving two equations: one with \( L \) and then one with \( U \).

4. The matrix \( A = \begin{bmatrix} 3 & 6 & 1 & 0 & 7 \\ 2 & 4 & 0 & 1 & 10 \\ 1 & 2 & 1 & -1 & -3 \\ 0 & 0 & 0 & 3 & 12 \end{bmatrix} \) has reduced row echelon form \( R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) Use \( R \) to determine the dimensions of the spaces \( \text{Col}(A) \), \( \text{Null}(A) \), \( \text{Row}(A) \), and \( \text{Null}(A^T) \).
(b) Find bases for the spaces \( \text{Col}(A) \), \( \text{Null}(A) \), and \( \text{Row}(A) \). The number of vectors in each basis set should be consistent with the dimensions you found in (a).

5. Classify each statement as true or false and give a brief justification of your answer.
(a) If \( A \) is a square matrix and \( Ax = 0 \) has a unique solution then the equation \( Ax = b \) is always consistent.
(b) The square matrix \( A \) is invertible if and only if \( \det A = 0 \).
(c) If \( b \) is a given nonzero vector, then the set of all solutions \( x \) to \( Ax = b \) is a subspace.
(d) If \( A \) is an \( m \times n \) matrix and \( n > m \) then the nullspace of \( A \) is not \( \{ 0 \} \).
(e) If \( A \) is an \( m \times n \) matrix then \( \dim \text{Null} A + \dim \text{Row} A = n \).
(f) The rank of a matrix \( A \) is equal to the nullity of \( A^T \).
(g) If \( A \) is an \( n \times n \) matrix and rank \( A < n \) then \( 0 \) is a root of the characteristic polynomial of \( A \).
(h) If \( \lambda \) is an eigenvalue of \( A \) with algebraic multiplicity \( r \) and \( W \) is the corresponding eigenspace then \( \dim W \) can take any value from 0 to \( r \).
(i) Every \( n \times n \) matrix with \( n \) distinct eigenvalues is diagonalizable.

6. In each case below let \( W \) be indicated set of vectors. Determine whether \( W \) is a subspace of \( \mathbb{R}^3 \). If it is, give \( \dim W \). If \( \dim W \geq 1 \) find a basis for \( W \).
(a) \( \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} \right\} \);
(b) \( \left\{ \begin{bmatrix} r \\ -8s \\ r+s \end{bmatrix} : r, s \in \mathbb{R} \right\} \);
(c) \( \left\{ \begin{bmatrix} r+s \\ -8(r+s) \\ 2r+2s \end{bmatrix} : r, s \in \mathbb{R} \right\} \);
(d) \( \left\{ \begin{bmatrix} r \\ -8s \\ r+s+1 \end{bmatrix} : r, s \in \mathbb{R} \right\} \);
(e) \( \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : r, s, t \in \mathbb{R} \right\} \);
(f) \( \left\{ \begin{bmatrix} r \\ -8r \\ 2r \end{bmatrix} : r = 0 \right\} \).
7. Let \( A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 3 & 4 \\ 1 & -2 & 1 & 2 \\ 3 & -3 & -2 & 1 \end{bmatrix} \).

(a) Evaluate \( \det A \) by a cofactor expansion along the first row.
(b) Evaluate \( \det A \) by a cofactor expansion along the second row.
(c) Evaluate \( \det A \) by row reduction of \( A \) to upper triangular form \( U \). (Don’t calculate \( \text{rref}(A) \).)

8. Let \( A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) be a \( 3 \times 3 \) matrix with row vectors \( a, b, \) and \( c \). Assume that \( \det A = 5 \).

(a) Find row operations that transform \( A \) into the matrix \( B = \begin{bmatrix} c + 3b \\ 2b \\ a \end{bmatrix} \). Then calculate \( \det B \).
(b) Let \( C = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{bmatrix} \). Find the determinant of the matrix \( AC^3A^T \).

9. (a) Find the eigenvalues and eigenvectors of the matrix \( A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} \).
(b) Find an invertible matrix \( P \) and diagonal matrix \( D \) such that \( A = PDP^{-1} \).

10. Let \( v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and let \( A = vv^T \). Note that \( A \) is a \( 3 \times 3 \) matrix.

(a) Show that \( v \) is an eigenvector of \( A \). What is the eigenvalue? (Hint: compute \( Av \) using the associative property of matrix multiplication.)
(b) Show that \( v \) is a basis for \( \text{Col} \ A \). (Hint: Show that each column of \( A \) is a multiple of \( v \).)
(c) What is \( \dim \text{Null} \ A \)?
(d) Find the characteristic polynomial of \( A \), the eigenvalues of \( A \), and their algebraic multiplicities.
(e) Show that \( A \) is diagonalizable (you don’t need to find all the eigenvectors).
(f) If \( A = PDP^{-1} \) with \( D \) diagonal, what are the diagonal entries of \( D \)?

11. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \).

(a) Find the characteristic polynomial of \( A \) and the eigenvalues of \( A \). Give the algebraic multiplicity of each eigenvalue.
(b) For each eigenvalue find a basis for the corresponding eigenspace.
(c) Determine whether or not \( A \) is diagonalizable.

12. Do the True-False questions from Sections 2.6, 3.1, 3.2, 4.1–4.3, 5.1–5.3 that are listed in the homework assignments.