Note: This problem set concentrates on material from the end of the course. For a complete review, you should also study the review problem sets for the two in-class exams. Please consider these earlier problem sets as implicitly included with this one. Particular topics that should be reviewed from earlier sets include: (i) Solving systems of linear equations, row operations, elementary matrices; (ii) The LU decomposition of a matrix; (iii) Inverses of matrices; (iv) Subspaces, finding bases for Col A, Row A, and Null A; (v) Determinants and characteristic polynomial of a matrix.

1. Let \( u \) and \( v \) be vectors in \( \mathbb{R}^n \).
   (a) State the Cauchy–Schwarz inequality and the triangle inequality for \( u \) and \( v \).
   (b) Prove the triangle inequality from the Cauchy–Schwarz inequality by calculating \( \|u + v\|^2 \).

2. Suppose that \( u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), \( v = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \), and that \( w \) is a vector in \( \mathbb{R}^3 \) with \( \|w\| = 5 \) and \( w \cdot u = 13 \).
   (a) Compute \( \|u\|, \|v\|, u \cdot v, \) and \( \|u + v\| \).
   (b) Show that the Cauchy-Schwarz and triangle inequalities are satisfied by \( u \) and \( v \).
   (c) Compute \( (u + 2w) \cdot (u - w) \).

3. Let \( V \) be the subspace of \( \mathbb{R}^3 \) spanned by the vector \( v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Let \( x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \).
   (a) Find the vector \( y \) that is the orthogonal projection of \( x \) onto \( V \). Then calculate \( z = x - y \) and check that \( z \perp V \).
   (b) Find a basis for \( V^\perp \) (the subspace of vectors orthogonal to \( V \)). (Hint: This is the null space of a \( 1 \times 3 \) matrix.)
   (c) Use part (b) and Gram-Schmidt to obtain an orthonormal basis \( \{q_1, q_2\} \) for \( V^\perp \).
   (d) Let \( z \) be the vector from (a). Then \( z \in V^\perp \), so \( z = c_1 q_1 + c_2 q_2 \) for suitable coefficients \( c_1, c_2 \). Give the general formula for these coefficients in terms of inner products, and use the formula to calculate the coefficients for this particular \( z \). Then check that \( z = c_1 q_1 + c_2 q_2 \).

4. Let \( A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \).
   (a) Give the dimensions of Row \( A \), Col \( A \), and Null \( A \).
   (b) Find orthonormal bases for Row \( A \), Col \( A \), and Null \( A \). Hint: One of these requires no calculation, one requires a small calculation, and one requires Gram-Schmidt.

5. Find a \( 3 \times 3 \) orthogonal matrix \( Q \) with first column \( \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \).
   Hint: There are some easy choices for columns 2 and 3.

6. True or false (four separate cases–justify your answer in each case). If a \( 4 \times 4 \) matrix \( A \) satisfies the following condition, it is diagonalizable:
   T F (a) the eigenvalues of \( A \) are 0, 1, 2, 3.
   T F (b) the characteristic polynomial of \( A \) is \( \lambda^2(\lambda - 1)(\lambda - 2) \);
   T F (c) the eigenvalues of \( A \) are 0, 1, and 2, and \( A \) has rank 2;
   T F (d) the eigenvalues of \( A \) are 0 and 2, and \( A \) is symmetric;

7. (a) Find the eigenvalues and eigenvectors of the matrix \( A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix} \).
   (b) Find an invertible matrix \( P \) and diagonal matrix \( D \) such that \( A = PDP^{-1} \).
8. A certain $3 \times 3$ matrix $A$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, and corresponding eigenvectors

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Use the formula $A = PDP^{-1}$ (for suitable $P$ and $D$) to find $A$.

(b) Let $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Use (a) to find coefficients $c_1$, $c_2$, $c_3$ so that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Then compute $A^n\mathbf{x}$ from this formula for $\mathbf{x}$ for arbitrary $n > 0$. What is a good approximation to $A^n\mathbf{x}$ for $n$ large?

9. Suppose that $A$ is a symmetric $n \times n$ matrix and that the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfy $A\mathbf{x} = 2\mathbf{x}$ and $A\mathbf{y} = 3\mathbf{y}$. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

10. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & 4 \end{bmatrix}$.

(a) Find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $A = \mathbf{vv}^T$. Show that $\mathbf{v}$ is an eigenvector for $A$ and find the eigenvalue.

(b) Calculate the nullity of $A$ and find a basis for the zero eigenspace of $A$. Check that $\mathbf{v} \perp \text{Null}(A)$ and explain why you know this without explicit calculation.

(c) Use (a) and (b) to find an orthonormal set of eigenvectors of $A$ which form a basis for $\mathbb{R}^3$.

(d) Find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that such that $A = QDQ^T$.

11. Classify each statement as true (T) or false (F). If your answer is T, give a brief proof showing that the statement is always true; if your answer is F, give a specific example for which the statement is not true.

T F (a) The null space of a matrix $A$ is the orthogonal complement of the column space of $A$.

T F (b) Every orthogonal matrix has null space $\{0\}$.

T F (c) If $P$ and $Q$ are orthogonal matrices then $P^TQ$ is an orthogonal matrix.

T F (d) If $A$ is an $n \times n$ matrix and $0$ is an eigenvalue of $A$ then $\text{Col}(A) \neq \mathbb{R}^n$.

T F (e) If $Q$ is an orthogonal matrix then $Q = Q^{-1}$.

T F (f) If $A$ is an $n \times n$ matrix then eigenvectors for distinct eigenvalues of $A$ are orthogonal.

12. Suppose that $W$ is a subspace of $\mathbb{R}^n$ of dimension $k$ and that $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ such that $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis for $W$.

(a) Any vector $\mathbf{u} \in \mathbb{R}^n$ has an expansion $\mathbf{u} = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$. Give a simple formula for the coefficients $c_j$ in terms of inner products.

(b) We know that any $\mathbf{u} \in \mathbb{R}^n$ can be written uniquely as $\mathbf{u} = \mathbf{w} + \mathbf{z}$, with $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$. Explain why $\mathbf{w} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k$.

(c) Let $C$ be the $n \times k$ matrix with columns $\mathbf{w}_1, \ldots, \mathbf{w}_k$. Then $W = \text{Col}(C)$. Show that $C^TC = I_k$. Then using your answers to (a) and (b), show that $P_W$, the orthogonal projection matrix onto $W$, is given by $P_W = CC^T$. (Recall that, in the notation of (b), $\mathbf{w} = P_W\mathbf{u}$.)

(d) Derive the result in (d) from the general formula for $P_W$ in terms of $C$.

13. Consider the data points $(-3, 9), (-1, 7), (0, 5), (4, 1)$ in the $(x, y)$ plane.

(a) The method of least squares for a straight line fit to this data minimizes a certain quantity. What is that quantity in this case? Give the answer explicitly; define any variables used.

(b) We obtain a solution by solving the normal equations $C^TC\mathbf{u} = C^T\mathbf{y}$. What is $C$ for the data above? What is $\mathbf{y}$? What is $\mathbf{u}$?

(c) Find the equation of the straight line which best fits this data.

14. Do the True-False questions from Sections 6.1 through 6.6 that are listed in the homework assignments.