Oral Qualifying Exam Syllabus

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I. PDE and Analysis

A. Partial Differential Equations
   i. Laplace’s Equation
      • Fundamental solutions including derivations
      • Green’s identities
      • Mean Value Theorem
      • Green functions and the Poisson kernel, including the Dirichlet problem on the unit ball
      • Uniqueness of solutions to the Dirichlet problem via the Maximum Principle
      • Perron’s existence method for the Dirichlet problem
   ii. Wave Equation
      • Characteristic surfaces (light cones)
      • Energy estimates
      • $C^2$ uniqueness via energy
      • Fundamental solutions (derivation for $n = 1$ and $n = 3$)
      • Duhamel’s Principle
      • Dispersive ($L^\infty$) estimate

B. Analysis
   i. Hilbert Spaces
      • Closed subspace decomposition
      • Self-duality
      • Bessel’s identity
      • Completeness
   ii. Elements of Fourier Analysis
      • Schwarz space as a Fréchet space
      • Properties of convolutions and Young’s Inequality
• Approximations of the identity (density of $C_0^\infty$ in $L^p$ and the $C^\infty$ Urysohn Lemma)
• Fourier Transform on $\mathbb{R}^n$:
  – Properties
  – Riemann-Lebesgue Lemma
  – Fourier inversion
  – Plancherel Formula

iii. Theory of Distributions
• Definition of convergence of test functions and continuity of functionals
• Derivatives and convolutions of distributions and test functions
• Density of $C_0^\infty$ in $\mathcal{D}'$
• Tempered distributions - examples and Fourier Transform

iv. $L^2$-Sobolev Spaces
• Definition of $H_s$
• Equivalent norms
• Properties, including density of $H_s$ in $H_t$ ($t < s$)
• Sobolev embedding theorem for $H_s$
• Rellich’s compactness theorem
• Definition of a localized $H_s$ space
• The elliptic regularity theorem for constant coefficient operators

v. Inequalities
• Hölder’s Inequality
• Minkowski’s Integral Inequality
• Sobolev Inequality applied to the hydrogen atom (stability of matter)
• Heisenberg’s Inequality (Uncertainty Principle)
II. Mathematical Physics

A. Newtonian Point Mechanics
   i. Newton’s equation of motion
      • Galilean invariance
      • The Lorentz Force law for test particles (test particle motion in a uniform electric and magnetic field)
   ii. Lagrangian formulation
      • Euler-Lagrange equations
      • Equivalence to Newton’s equation of motion
      • The 2-body problem for attractive/repulsive 1/r potential
      • Noether’s Theorem (symmetry and conservation laws)
   iii. Hamiltonian formulation
      • Legendre transformations
      • Equivalence to Newton’s equation of motion

B. Einsteinian Point Mechanics
   i. Changes with respect to Classical Physics
      • Poincaré Group
      • 4-vectors and invariants (proper time, wave operator etc.)
   ii. Maxwell’s equations for fields given charges/currents
      • Derivation of wave equation in Lorentz gauge
      • Gauge invariance
      • Covariant form of Maxwell’s equations

C. Quantum Mechanics
   i. Schrödinger’s Equation
      • The (non-relativistic) hydrogen atom as a 1-body problem (Pauli’s wave equation)
      • N-body Schrödinger Equation
      • Dirac Equation
References


