This exam will be given over two days, in two three hour sessions. Each session will consist
of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is
to ensure that all students at least attempt a range of questions, but one area of weakness
should not be overly magnified.
First Day – Part I: Answer each of the following three questions.

1. Let $X = \mathbb{R}$ and let $\tau = \{ G \subset X \mid X \setminus G \text{ is countable} \} \cup \{ \emptyset \}$. Show
   i) $\tau$ is a topology on $X$;
   ii) in the topological space $(X, \tau)$ the collection $\mathcal{U}_0$ of neighborhoods of 0 does not have a countable base.

2. Use contour integration to compute
   \[
   \int_0^\infty \frac{dx}{x^4 + 1}.
   \]

3. Suppose that
   \[
   A = \begin{bmatrix}
   0 & 1 & 1 \\
   1 & 0 & 1 \\
   1 & 1 & 0
   \end{bmatrix}.
   \]
   Find a $3 \times 3$ orthogonal matrix $U$ such that $U^{-1}AU$ is diagonal.
First Day – Part II: Answer three out of the following six questions.

4. a) How many Sylow 5-subgroups does the alternating group $A_5$ have?  
b) How many Sylow 5-subgroups does the alternating group $A_6$ have?  
c) How many Sylow 5-subgroups does a simple group of order 360 have?  
(Hint: use part b).)

5. Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on $\mathbb{R}$, and let $p > 1$. Assume  
i) each $f_n$ is in $L_p$ and $f_n \longrightarrow f$ in $L_p$;  
ii) $g_n \longrightarrow g$ a.e.;  
iii) $|g_n| \leq M < \infty$ for each $n$.  
Show that $g_nf_n \longrightarrow gf$ in $L_p$.

6. Consider the topology on the set of integers $\mathbb{Z}$ for which $\{n+7^k\mathbb{Z}|k \in \mathbb{Z}\}$ is a basis for the neighborhoods of $n \in \mathbb{Z}$.  
i) Show that this is a Hausdorff topology;  
ii) Show there is a sequence $\{n_j\}$ in $\mathbb{Z}$ such that $(n_j)^2 \longrightarrow 2$ in this topology.

7. Consider the spiral $S$ in the plane which in polar coordinates has the equation $r = \theta$ for $\theta \geq 0$. Let $G = \mathbb{C} \setminus S$. Show there is a holomorphic function $L : G \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z^2$ for all $z \in G$. 
8. Let \( \mathbb{Z}_2 \) be the field of two elements. Consider the matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

over \( \mathbb{Z}_2 \).

a) Show that the characteristic polynomial of \( A \) factors as a product of linear polynomials over \( \mathbb{Z}_2 \).

b) Find a non-singular matrix \( P \) over \( \mathbb{Z}_2 \) so that \( P^{-1}AP \) is in Jordan form.

9. Let \( \mathbf{v}(x,t) \) be a \( C^\infty \) vector field and \( \rho(x,t) \) a \( C^\infty \) real-valued function defined for \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \). Assume that for any ball \( B \subset \mathbb{R}^3 \)

\[
\frac{d}{dt} \int \int \int_B \rho \, dx = - \int \int_{\partial B} \rho \mathbf{v} \cdot \mathbf{n} \, dA
\]

holds for all \( t \) where \( \mathbf{n} \) is the exterior unit normal on \( \partial B \) and \( dA \) is the element of surface area. Show that

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

for all \((x,t)\).
Second Day – Part I: Answer each of the following three questions.

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Assume that \( f(x) \to L \) as \( x \to \infty \).
   a) Show that if \( \lim_{x \to \infty} f'(x) \) exists then \( \lim_{x \to \infty} f'(x) = 0 \).
   b) Is a) true if “\( \lim_{x \to \infty} f'(x) \) exists” is replaced by “\( f'(x) \) is bounded”?

2. For a ring \( R \) let \( M_n(R) \) denote the ring of \( n \times n \) matrices with entries in \( r \). Prove that if \( R \) is a ring with multiplicative identity 1, then every two-sided ideal of \( M_n(R) \) is of the form \( M_n(I) \), where \( I \) is an ideal of \( R \).

3. Let \( m \) be the Lebesgue measure on \( \mathbb{R} \). Let \( A \subset \mathbb{R} \) be a measurable set such that there exists a number \( b \) with \( 0 < b < 1 \) and
   \[
m(A \cap I) < bm(I)
   \]
   for all open intervals \( I \). Show that \( m(A) = 0 \).
Second Day – Part II: Answer three of the following six questions.

4. Let \( \{f_n\} \) be a sequence of Lebesgue measurable functions on the interval \((0,1)\) such that

i) \( \sup \int_0^1 |f_n| \, dx < \infty \)

ii) \( f_n \rightharpoonup 0 \) in measure.

Show that

\[ \int_0^1 \sqrt{|f_n|} \, dx \to 0. \]

5. Let \((X, \tau)\) be a compact metric space. Prove that if \( \{U_1, U_2, \ldots, U_k\} \) is an open cover of \( X \), then there exists a closed cover \( \{C_1, C_2, \ldots, C_k\} \) with \( C_i \subset U_i \) for each \( i \).

6. Let \( Q = \{ z \in \mathbb{C} | \text{Re} \, z > 0, \text{Im} \, z > 0 \} \) and let \( E = \{ z \in \mathbb{C} | |z| < 1 \} \). Give explicitly a holomorphic function \( f : Q \to E \) which is one-to-one and onto.

7. Let \( \mathbb{J} \) denote the set of Gaussian integers; that is,

\[ \mathbb{J} = \{ \alpha + \beta i | \alpha, \beta \in \mathbb{Z} \}. \]

Let \( u \) and \( v \) be nonzero Gaussian integers, and let \( \bar{\gamma} \) denote the conjugate of the complex number \( \gamma \).

a) Show that there exist Gaussian integers \( x, y \) such that \( uv = xv\bar{v} + y \) and \( y\bar{y} < (v\bar{v})^2 \).

b) Show that \( \mathbb{J} \) is a Euclidean domain whose degree function is complex absolute value squared.

c) Find the greatest common divisor (in \( \mathbb{J} \)) of the two Gaussian integers \( 7 - 6i \) and \( 7 - 11i \).
8. For what values of $\lambda$ will the solution $x(t)$ to
\[
x'(t) = x^3(t) + \lambda x^2(t)
\]
exist for all $t \geq 0$ and be uniformly bounded. Justify your answer fully.

9. Find two $4 \times 4$ matrices $A$ and $B$ such that:
   i) $A$ and $B$ have the same characteristic polynomial;
   ii) $A$ and $B$ have the same minimal polynomial;
   iii) $A$ and $B$ are not similar.