This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.
First Day – Part I: Answer each of the following three questions.

Question 1. Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces, and \(f : X_1 \rightarrow X_2\) a continuous surjective map such that \(d_1(p, q) \leq d_2(f(p), f(q))\) for every pair of points \(p, q \in X_1\).

(a) If \(X_1\) is complete, must \(X_2\) be complete? Give a proof or a counterexample.
(b) If \(X_2\) is complete, must \(X_1\) be complete? Give a proof or a counterexample.

Question 2. Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be an integrable function, and let \(g : \mathbb{R} \rightarrow \mathbb{R}\) be a bounded measurable function which is continuous at each \(x\) outside a set \(A\) of Lebesgue measure zero. Show that \(F(t) = \int_{-\infty}^{\infty} f(x)g(xt)dx\) is a continuous function for \(t \neq 0\).

Question 3. Let \(G\) be a group and \(H\) and \(K\) subgroups such that \(H\) has finite index in \(G\). Prove that \(H \cap K\) has finite index in \(K\).
First Day – Part II: Answer three out of the following six questions.

**Question 4.** Suppose that \( f(x) \) is a continuous real-valued function with domain \( \mathbb{R} \) which is differentiable for all \( x \neq 0 \).

(a) If \( \lim_{x \to 0} f'(x) \) exists, show that \( f'(0) \) exists.

(b) If \( \lim_{x \to 0} f'(x) \) need not exist, must \( f'(0) \) exist? Prove or give a counterexample.

**Question 5.** Prove or disprove: there is a real \( n \times n \) matrix \( A \) such that
\[
A^2 + 2A + 5I = 0
\]
if and only if \( n \) is even. (Here \( I \) denotes the \( n \times n \) identity matrix).

**Question 6.** Let \( f \) be an analytic function that maps the open unit disk \( D \) into itself and vanishes at the origin.

(a) Prove that \( |f(z) + f(-z)| \leq 2|z|^2 \) in \( D \).

(b) Prove that the inequality in 6(a) is strict, except at the origin, unless \( f \) has the form \( f(z) = \lambda z^2 \) for some \( \lambda \) a constant of absolute value one.

**Question 7.** Let \( f(x) = x^5 + 2x^3 + 2x^2 + x - 3, g(x) = x^4 + 3x^2 + 2x + 3 \). Prove that there is an integer \( d \) such that the polynomials \( f(x) \) and \( g(x) \) have a common root in the field \( \mathbb{Q}[\sqrt{7}] \). What is \( d \)?

**Question 8.** Let \( \{f_n\}_{n=1}^\infty \) be a sequence of real-valued \( C^1 \) functions on \([0, 1]\) such that, for all \( n \),
\[
|f'_n(x)| \leq x^{-1/2} \quad \text{for} \quad (0 < x \leq 1), \quad \text{and} \quad \\
\int_0^1 f_n(x)dx = 0.
\]
Prove that the sequence has a subsequence that converges uniformly on \([0, 1]\).

**Question 9.** Let \( V \) be a finite-dimensional linear subspace of \( C^\infty(\mathbb{R}) \) (the space of complex-valued, infinitely differentiable functions). Assume that \( V \) is closed under \( D \), the operator of differentiation (i.e., \( f \in V \Rightarrow Df = f' \in V \)). Prove that there is a constant coefficient differential operator
\[
L = \sum_{k=0}^n a_k D^k
\]
such that \( V \) consists of all solutions of the differential equation \( Lf = 0 \).
Second Day – Part I: Answer each of the following three questions.

Question 1. Evaluate
\[ \int_C \frac{e^z}{z(2z+1)^2} \, dz \]
where \( C \) is the unit circle oriented counterclockwise.

Question 2. Let \( A_n \) be a sequence of Lebesgue measurable subsets of \([0, 1]\), and \( A_\infty = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n \) (the set of points that belong to infinitely many of the sets \( A_n \)).

(a) Prove that \( \sum_{n=1}^{\infty} m(A_n) < \infty \) is a sufficient condition for \( m(A_\infty) = 0 \).

(b) Prove that \( \lim_{n \to \infty} m(A_n) = 0 \) is a necessary condition for \( m(A_\infty) = 0 \).

Which of these two conditions remain valid if we allow \( A_n \) to be arbitrary Lebesgue measurable subsets of \( \mathbb{R} \)?

Question 3. Let \( A \) and \( B \) be two diagonalizable \( n \times n \) complex matrices such that \( AB = BA \). Prove that there is a nonsingular \( n \times n \) matrix \( P \) such that both \( P^{-1}AP \) and \( P^{-1}BP \) are diagonal matrices.
Second Day – Part II: Answer three of the following six questions.

**Question 4.** A standard theorem states that a continuous real-valued function on a compact set is bounded. Prove the converse: if $K$ is a subset of $\mathbb{R}^n$, and if every continuous real-valued function on $K$ is bounded, then $K$ is compact.

**Question 5.** Let $p, q, r$ be continuous real-valued functions on $\mathbb{R}$ with $p(t) > 0$ for all $t \in \mathbb{R}$. Prove that there exist a continuously differentiable function $a(t)$ and a continuous function $b(t)$ such that the differential equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$

has exactly the same solutions as the equation

$$[a(t)x'(t)]' + b(t)x(t) = 0.$$  

**Question 6.** Let $F$ be a field. Prove that every finite subgroup of the multiplicative group of nonzero elements of $F$ is cyclic.

**Question 7.** Let $I$ denote the ideal in $\mathbb{Z}[X]$, the ring of polynomials with coefficients in $\mathbb{Z}$, generated by $x^3 + x + 1$ and 5. Is $I$ a prime ideal? Justify your answer.

**Question 8.** Let $O$ be open and $f : O \to \mathbb{C}$ be holomorphic and one-to-one. Show that for any $z_0 \in O$, the level curves $\Gamma_1 = \{ z : Re f(z) = Re f(z_0) \}$ and $\Gamma_2 = \{ z : Im f(z) = Im f(z_0) \}$ intersect at right angles.

**Question 9.** Let $\mathbb{R}^2$ represented as $2 \times 1$ column vectors be equipped with the Euclidean metric: $d(x, y) = ||x - y||$ where $|| \cdot ||$ is the Euclidean norm. Let $T$ be an isometry (=distance preserving map) of $\mathbb{R}^2$ into itself. Prove that $T$ can be represented as

$$T(x) = a + Ux,$$

where $a$ is a vector in $\mathbb{R}^2$ and $U$ is an orthogonal matrix.