An explicit construction of Parisi landscapes in finite dimension

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Disordered systems and landscapes

- Phenomenological description of glassy systems (glasses, proteins, spin-glasses): rugged energy landscapes

- Long (Goldstein 1969), useful but somewhat misleading tradition (one dimensional cartoons...) – coherence length scale implied

- Classification of random landscapes (lessons from Spin-glasses)
  - SK model and Parisi's full-RSB: complex, hierarchical landcapes – valleys within valleys, ultrametricity
  - REM and 1-step RSB: random (golf course) landscape on the hypercube

Universality: Extreme value statistics

- Low temperature physics of disordered systems: statistics of deep energies

- $M \gg 1$ IID Gaussian variables, $\epsilon_{\max}(M) = \sigma\sqrt{2\ln M}\left[1 + \frac{u}{2\ln M}\right]$

- Extreme value distributions for IID
  - Exponential variables: Gumbel $- G(u) = \exp[-u - \exp[-u]]$
  - Bounded variables: Weibull $- H(u) = \mu u^{\mu-1}\exp[-u^{\mu}]$
  - Power-law variables: Fréchet $- F(u) = \mu \exp[-u^{-\mu}]/u^{1+\mu}$

Extreme value statistics

- The Random Energy Model: ([Derrida], and also [Cabibbo, Lifshitz-Pastur]) energy of states are IID
  - $M = 2^N$, $\sigma = \sqrt{N} \rightarrow \epsilon_{\text{max}} \sim N$ and $O(1)$ energy gap: non trivial thermodynamics and glass transition
  - Below T_c the Boltzmann measure is localized on a finite number of states: $Y = \sum p_i^2 = 1 - T/T_c$ – random condensation
  - (1-Step) Replica Symmetry Breaking encodes the Gumbel statistics of low energy states

Random potential (finite dimensions, short range)

- One particle in a short-range correlated Gaussian random potential $V(\mathbf{r})$ in N dimensions:

$$V_{\mathrm{max}}(R) \sim \sigma\sqrt{\ln R^N}$$

- Not strong enough to compete with entropy: $S \sim \ln R^N \to$ always in the high temperature (delocalized) phase

- Diffusive motion at any T: $D \sim \exp(-\sigma^2/2T^2)$

Random potential (finite dimensions, long range)

- One particle in a long-range correlated Gaussian random potential $V(\mathbf{r})$ in N dimensions:

  $\langle V(\mathbf{r})V(\mathbf{r}')\rangle = Ng^2|\mathbf{r}-\mathbf{r}'|^{2\theta}, \quad (\theta > 0) \quad \to V_{\text{max}}(R) \sim \sqrt{N}gR^{\theta}$

- Always beats entropy $S \sim \ln R^N$ $\to$ always in the low temperature (localized) phase

- Example: Exactly soluble Sinai (random force) model in $N = 1$ dimension ($\theta = 1/2$) $\to$ logarithmic diffusion ([Fisher-Le Doussal-Monthus])

Random potential (marginal)

- A logarithmically correlated random potential in N dimensions (e.g. thermal interface in 2d):

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = N\left[f_0 - g^2 \ln(|\mathbf{r}-\mathbf{r}'|^2 + a^2)\right] \qquad \theta = 0$$

- Simple argument: $V_{\text{max}} \sim g\sqrt{N \ln R}\sqrt{\ln R^N} \sim Ng \ln R$ matches $TN \ln R$ at $T_c = g$

- Using RG, the free energy given by a REM-like expression with a REM freezing transition at $T_c = g$ indep of N ([Carpentier-Le Doussal]) – matches exact results at $N = \infty$ ([Fyodorov-Sommers])

- Describes the energy landscape of the Directed Polymer on a tree ([Derrida-Spohn])

Random potential (marginal)

- A dynamical transition at T_c ([Castillo-Le Doussal]) where aging sets in

- Building block of the Bacry-Muzy-Delour multifractal random walk (turbulence, finance)

Extreme distribution for log correlated variables

- Lowest energy states have a non Gumbel distribution: a conjecture ([Fyodorov-JPB])

$$P(u) = -\frac{d}{du}\left[2e^{\frac{u}{2g}}K_1\left(2e^{\frac{u}{2g}}\right)\right]; \qquad V = -Ng \ln R + u$$

A multiscale logarithmic potential

- Logarithmically correlated random potential with several scales: $V(\mathbf{r}) = \sum_{i=1}^{K} V_i(\mathbf{r})$ with

$$\left\langle V_i(\mathbf{r}_1)\, V_j(\mathbf{r}_2) \right\rangle_V = \delta_{i,j} N\, f_i\left(\frac{1}{2N}(\mathbf{r}_1 - \mathbf{r}_2)^2\right) , \qquad f_i(u) = -g_i^2 \ln\left(u + R^{2\nu_i}\right)$$

  with increasing $0 \leq \nu_i \leq 1$ – separation of length-scales in the $R \to \infty$ limit

- In the continuum limit

$$g^2 \ln(r^2 + a^2) \longrightarrow \int_0^1 \rho(\nu) g^2(\nu) \ln\left(r^2 + R^{2\nu}\right) d\nu,$$

- Exact results for the free-energy in the $N \to \infty$ limit ([Fyodorov-JPB]) but expected to be valid at finite N as well

A multiscale logarithmic potential

- For a discrete spectrum $g^2(\nu)\rho(\nu) = \sum_{i=1}^{K} g_i^2\,\delta(\nu - \nu_i)$, the model has exactly the GREM free energy in the limit $R \to \infty$.

- For each temperature $T_p = \sum_{i=p}^{K} g_i^2$ the system freezes within blobs of size R^{ν_p} *à la* REM, with $Y_2(p) = 1 - T/T_p$

- The system first freezes at $T_c = T_1$ on the largest scale

- The last freezing transition takes place at $T_{\mathrm{min}} = T_K$ on scale R^0

A multiscale logarithmic potential

- In the continuum limit: an explicit construction of Parisi landscapes in *finite* Euclidean dimensions in terms of GTI processes; infinite sequence of freezing transitions

- Multifractal Boltzmann measure

$$Y_q = \int_V p_\beta^q(\mathbf{r})\, d\mathbf{r} \sim V^{-\tau_q}$$

  leads to $f(\alpha)$ that are generically singular when $f(\alpha_\pm) = 0$

- Multiscale dynamical freezing all levels such that $T < T_p$ age (concerning large length scales), whereas small length scales, such that $T > T_p$, are still stationary: temperature as a microscope

A multiscale logarithmic potential

- Generalized multifractal random walk with epoch dependent multifractal spectrum

Conclusion, open problems

- An explicit construction of Parisi landscapes in finite Euclidean dimensions, of pedagogical interest (not so exotic after all...)

- Open problem: how/does the RSB/FRG algebra extend to Fréchet (/Weibull) distributions? What is the structure of the low temperature mean-field power-law spin-glasses and the solution of pinned manifolds with power-law disorder?