Random Band Matrices and SUSY Statistical Mechanics

Joint work with:

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a) **RBM** may be studied via **SUSY** Statistical Mechanics model - F. Wegner, K. Efetov

b) **Main Result:** Simpler **SUSY Hyperbolic Sigma** model has a delocalized phase with Q-diffusion in 3D.

c) Model equivalent to Random Walk in correlated random environment.

d) Analysis relies on **SUSY** Ward identities.

SUSY hyperbolic sigma model – Zirnbauer (1991).

This model is expected to have many of the features of the exact SUSY models of Random Band Matrices.

Hyperbolic Sigma model: spins are 2×2 matrices:

$$S_j = T_j^{-1}\sigma_3 T_j \qquad S_j^2 = I$$

where $T \in SU(1,1)$ and we use Haar measure.

T_j preserves $|z_1|^2 - |z_2|^2$

Effective action is convex. Thus there is **no phase transition**.

SUSY Hyperbolic Sigma model

Metric:

$$-dz^2 + dx^2 + dy^2 + 2d\overline{\psi}d\psi$$

Sigma model constraint:

$$-z^2 + x^2 + y^2 + 2\overline{\psi}\psi = -1$$

In **horospherical** coordinates:

$$t_j,\ s_j \in \mathbb{R},\ \ and\ \ \overline{\psi}_j, \psi_j\ \ Grassmann$$

$$\mathbf{A}_\epsilon = \mathbf{\Sigma}'(\mathbf{cosh}(\mathbf{t_j} - \mathbf{t_{j'}}) - \mathbf{1}) + \frac{\epsilon}{\beta}\sum_{\mathbf{j}\in\mathbf{\Lambda}} \mathbf{cosh}\ \mathbf{t_j}$$

$$+ \mathbf{\Sigma}'\mathbf{e}^{(\mathbf{t_j}+\mathbf{t_{j'}})}\Big[\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{s_j} - \mathbf{s_{j'}})^{\mathbf{2}} + (\overline{\psi}_\mathbf{j} - \overline{\psi}_{\mathbf{j'}})(\psi_\mathbf{j} - \psi_{\mathbf{j'}})\Big]$$

Σ' is over nearest neighbors in $\mathbb{Z}^d \cap \Lambda$.

After integrating over $s, \overline{\psi}, \psi$ fields we obtain an effective **nonconvex** bosonic action $\mathbf{B}_\epsilon(\mathbf{t})$:

$$\exp[-\beta \mathbf{B}_\epsilon(\mathbf{t})] \equiv \exp[-\beta[\mathbf{\Sigma}' \, \mathbf{cosh} \, (\mathbf{t_j} - \mathbf{t_{j'}}) - \mathbf{1}]]$$

$$\cdot \, \mathbf{det^{1/2}}[\beta \mathbf{D}(\mathbf{t}) + \epsilon \mathbf{e^t}] \cdot \mathbf{e}^{-\epsilon \mathbf{\Sigma}[\mathbf{cosh}(\mathbf{t_j}) - \mathbf{1}]}.$$

$$0 \leq \mathbf{D(t)} = \sum_{\alpha=1}^{d} (\nabla_\alpha^* e^{t_j + t_{j'}} \nabla_\alpha)$$

$$(f, \; \mathbf{D}(t) f) = \Sigma' \, e^{t_j + t_{j'}} \, (f(j) - f(j'))^2$$

$$< F(t) > (\beta) \equiv \int F(t) \exp[-\beta B_\epsilon(t)] \prod_j e^{-t_j} \, dt_j$$

$$Z(\beta, \epsilon) = < 1 > (\beta) = 1$$

Analogue of Q-Transport is RWRE:

$$< |(E + i\epsilon - H)^{-1}(x,y)|^2 >$$

$$\cong < e^{t_x + t_y}[\beta \mathbf{D(t)} + \epsilon e^{t}]^{-1}(x,y) > (\beta)$$

Recall $\mathbf{D}(t) = \sum_{\alpha=1}^{d} (\nabla_\alpha^* e^{t_j + t_{j'}} \nabla_\alpha)$

$\beta(E) = \rho(E)^2 W^2$ Band width W

$= \rho(E)^2 \frac{1}{\lambda^2}$ Schrödinger

Here $\lambda =$ the strength of disorder.

Theorem(Zirnbauer) Localization in **1 Dim**:

$$< e^{t_0+t_x}[\beta \mathbf{D}(t) + \epsilon e^{t}]^{-1}(0,x) > (\beta)$$

$$=< [\beta \tilde{\mathbf{D}}(t) + \epsilon e^{-t}]^{-1}(0,x) > (\beta)$$

$$\cong \epsilon^{-1} e^{-|x|/\beta}$$

$$\tilde{\mathbf{D}} = e^{-t}\mathbf{D}e^{-t} = -\Delta + V(t) \geq 0.$$

The **saddle point** $t = t_s$ minimizes $\mathrm{B}_\epsilon(\mathrm{t})$:

$$\begin{aligned} \epsilon e^{-t} &= 1/\beta && 1Dim - localization \\ \epsilon e^{-t} &= e^{-\beta} && 2Dim - localization \\ t &\cong 0 && 3Dim\ -\ \beta\ large - Diffusion \\ \epsilon e^{-t} &\sim 1 && 3Dim\ -\ \beta\ small - localization \end{aligned}$$

In all dimensions key problem to control fluctuations in t.

Main Theorem (Di-Sp-Zi):

In **3D** for **large** β :

$$< e^{t_x + t_y} [\beta \mathbf{D}(t) + \epsilon e^{t}]^{-1} (x, y) > (\beta)$$

$$\sim \frac{1}{-\beta^* \Delta + \epsilon} (x, y)$$

as quadratic forms where β^* is effective diffusion constant.

Thus in 3D, if β is large the walk is transient.

Analogous to problems for edge reinforced random walk!

Key to Proof: SUSY Ward Identities

$\mathbf{0SP(2|2)}$ invariance implies $Z(\beta, \epsilon) = 1$

More generally if $F(t, s, \overline{\psi}, \psi)$ is $\mathbf{0Sp(2|2)}$ invariant, then

$$< F >_{SUSY} (\beta, \epsilon) = F(0)$$

Example:

$$F = cosh(t_0 - t_\ell)$$

$$+ e^{t_0 + t_\ell}\Big[\frac{1}{2}(s_0 - s_\ell)^2 + (\overline{\psi}_0 - \overline{\psi}_\ell)(\psi_0 - \psi_\ell)\Big]$$

is $\mathbf{0SP(2|2)}$ invariant and $F(0) = 1$ therefore

$$< F^m >_{SUSY} = 1$$

Let $\ell = 1$ and integrate out $s, \overline{\psi}, \psi$ fields

$$1 =< e^{\widetilde{\beta}(cosh(t_0 - t_1) - 1)}(1 - \widetilde{\beta} G_1(t)) > (\beta)$$

Claim $\mathbf{G_1(t) \leq 1/\beta}$. *Hence*

$$< e^{\widetilde{\beta}(cosh(t_0 - t_1) - 1)} > \leq (1 - \widetilde{\beta} / \beta)^{-1}$$

Thus n.n. fluctuations are suppressed

$$Prob\ (|t_0 - t_1| \geq a) \leq e^{-\widetilde{\beta}(cosh(a) - 1)}$$

Use **induction** on ℓ to control longer scales.

Happy Birthday

Edouard and Giorgio