Diffusion in a 3D SUSY σ - model

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Outline of Talk

A) Motivation: Study time evolution of quantum particle in a random environment using **Statistical Mechanics**.

Advantage: Saddle point analysis + Symmetries

B) Spectral properties algebraically equivalent to correlations in a Supersymmetric Statistical Mechanics Model:

Hyperbolic, $SU(1,1|2)$ symmetry,
Spin = 4x4 super-matrix,
Partition function $\equiv 1$.

C) Result: Diffusion for a simpler 3D SUSY model:

Model due to Zirnbauer, has hyperbolic supersymmetry.

Expected to reflect a Localization-Diffusion transition.

Equivalent to random walk in a *correlated* random environment.

D) Key Ideas of proof:

Estimate fluctuations of the environment
using SUSY Ward Identities.

Estimates on **nonuniformly** elliptic equations.

E) Relation to Edge Reinforced Random Walk?

This is *history* a dependent random walk which favors edges it has visited in the past.

It is also equivalent to a random walk in a *correlated* random environment (Diaconis).

Localization in 1D + partial results in 2D (Merkl, Rolles) using a Mermin-Wagner type argument to estimate fluctuations of the environment.

Quantum Green's Function

Let **H** denote a *random* band matrix or Schrödinger operator.

$$G_\epsilon(E;x,y) \equiv (\mathbf{H} - E + i\epsilon)^{-1}(x,y).$$

Q-Diffusion Conjecture in 3D:

$$< |G_\epsilon(E;x,y)|^2 > \cong \frac{\rho(E)}{-D\Delta + \epsilon}(x,y) \approx C\,|x-y|^{-1}$$

$\rho(E) \; = \; Im < G_\epsilon(E;0,0) >=$ density of states

$D = D(E) > 0$ is diffusion constant.

Localization :

$$< |G_\epsilon(E;x,y)|^2 > \cong \epsilon^{-1}\, e^{-|x-y|/\ell}$$

SUSY Hyperbolic Sigma model

Sigma constraint: $-z_j^2 + x_j^2 + y_j^2 + 2\overline{\psi_j}\psi_j = -1.$

Action **A** in *horospherical* coordinates:

$$t_j,\ s_j \in \mathbb{R}, \ \ \textit{and} \ \ \overline{\psi}_j, \psi_j \ \ \textit{Grassmann}$$

$$\mathbf{A}_\epsilon(\mathbf{t}, \mathbf{s}, \psi) \equiv \boldsymbol{\Sigma}_{\mathbf{j}\sim\mathbf{j}'}(\mathbf{cosh}(\mathbf{t_j} - \mathbf{t_{j'}}) - \mathbf{1}) + \frac{\epsilon}{\beta}\sum_{\mathbf{j}} \mathbf{cosh}\ \mathbf{t_j}$$

$$+ \sum_{\mathbf{j}\sim\mathbf{j}'} \mathbf{e}^{(\mathbf{t_j}+\mathbf{t_{j'}})}\left[\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{s_j} - \mathbf{s_{j'}})^{\mathbf{2}} + (\overline{\psi}_{\mathbf{j}} - \overline{\psi}_{\mathbf{j'}})(\psi_{\mathbf{j}} - \psi_{\mathbf{j'}})\right]$$

$$Z = \int e^{-\beta A_\epsilon(t,s,\psi)} \prod_j e^{-t_j} dt_j ds_j d\psi_j d\overline{\psi_j} \equiv 1$$

Elliptic generator **D** with random conductances

For $t_j \in \mathbb{R}$, define $\mathbf{D(t)}$ via the quadratic form:

$$(f, \mathbf{D}_\beta(t)\, f) = \beta \sum_{j \sim j'} e^{t_j + t_{j'}} \, (f(j) - f(j'))^2.$$

D is the generator of *random walk in the random environment* of the t field. **D** is **not** uniformly elliptic.

If $e^{(t_j + t_{j'})} \sim 1$ then there is **diffusion**.

However, if $< e^{t_j/2} >_{\beta,\epsilon} << 1$, then conductance goes to zero and **localization** occurs. Known in 1D and expected in 2D

Relation to Quantum-Green's Function

$$< |G_\epsilon(E; x, y)|^2 >_{RM} \cong < s_x e^{t_x} s_y e^{t_y} >_{SUSY} (\beta, \epsilon)$$

$$= < [\beta\, \mathbf{D}\ (t) + \epsilon e^{t}]^{-1}(x, y) e^{t_x + t_y} >_{SUSY} (\beta, \epsilon)$$

$$= < [-\beta\ \Delta + V(t) + \epsilon e^{-t}]^{-1}(x, y) >_{SUSY} (\beta, \epsilon)$$

$\beta \sim$ density of states $\rho(E)$, times band width.

Main Theorem: "Diffusion" in 3D (Di-Sp-Zi):

For β **large**, local conductance $e^{t_j + t_{j'}} \sim 1$, hence :

$$< e^{t_0 + t_x}[\beta \mathbf{D}(t) + \epsilon e^t]^{-1}(0, x) >_{SUSY} (\beta)$$

$$\approx (-\beta\Delta + \epsilon)^{-1}(0, x) \approx C|x|^{-1}$$

So, the evolution is **"Diffusive"**.

Conjecture: For **small** β, **localization** occurs in 3D:
$< e^{t_j/2} > \to 0$ as $\epsilon \to 0$, conductance vanishes.

Ward Identities are Key to Proof

To estimate the conductance, we first bound the *fluctuations* of the t field,

$$< cosh^m(t_0 - t_\ell) > (\beta, \epsilon) \leq Const.$$

by induction on $|\ell|$ and Ward identities:

Example: For $\ell \in \mathbb{Z}^d$, *let*

$$F_\ell = cosh(t_0 - t_\ell) + \; e^{t_0 + t_\ell}\left[\frac{1}{2}(s_0 - s_\ell)^2 + (\overline{\psi}_0 - \overline{\psi}_\ell)(\psi_0 - \psi_\ell)\right]$$

$$Then \;\; < F_\ell^m > = 1 \;\; all \; m \geq 1.$$

After integrating over the ψ and s variables

The Ward identity $< F_\ell^m >= 1$ yields:

$$< cosh^m(t_0 - t_\ell)(1 - \frac{m}{\beta} G_\ell(t)) >= 1$$

where G_ℓ is the Green's function of $D(t)$.

If $0 \le G_\ell(t) \le \mathbf{C}$, then for $\mathbf{C}m < \beta$

$$< cosh^m(t_0 - t_\ell) > \le (1 - \frac{\mathbf{C}m}{\beta})^{-1}$$

Thus t - fluctuations at scale ℓ are small for large β.

Conclusions and open Problems

Quantum Dynamics $\approx$ SUSY hyperbolic model $=$ RWRE.

In our Hyperbolic SUSY model, localization and diffusion are revealed using a simple **saddle** analysis. But **fluctuations** need to be **estimated**. We do this in 3D for large β.

Once the Ward identities are established, analysis is classical: Induction on length scales $+$ estimates on elliptic Greens functions.

Problem: Prove Localization in 2D and Phase Transition in 3D.

Happy Birthday Yoseph!