RG fixed point stability and decay of correlations

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References

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J. Zinn-Justin, *Transitions de phase et groupe de renormalisation*, chap. 11, CNRS Editions et EDP Sciences (Les Ulis 2005), English version *Phase transitions and renormalization group*, Oxford Univ. Press (2007).

The property that the general β-function, at leading orders in the ε-expansion, defines a gradient flow has first been noticed by

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Abstract

Critical properties of a large class of continuous phase transitions can be inferred from a renormalization group analysis and are governed by the IR fixed points of ϕ^4-like field theories.

Within the framework of the ε-expansion, we show quite generally that the stablest fixed point corresponds to the fastest decay of correlations.

We conjecture that this property must be generally true.

A systematic survey of models for which non-perturbative results have been obtained, has comforted the conjecture.

The $O(N)$ fixed point below four dimensions

The critical properties of the simplest class of continuous phase transitions are described by an N-component field ϕ and the $O(N)$ symmetric $(\phi^2)^2$ statistical field theory, corresponding to the Hamiltonian

$$\mathcal{H}(\phi)=\int \mathrm{d}^d x\left[\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}r\phi^2+\tfrac{1}{4!}g\left(\phi^2\right)^2\right].$$

Below four dimensions a non-Gaussian IR fixed point $(g=g^*>0)$ is then found, which is stable against the Gaussian fixed point $(g=0)$.

Parametrizing, as usual, the field dimension as $2d_\phi=d-2+\eta$, such that for the Gaussian fixed point the exponent $\eta=0$, one derives from the spectral representation of the two-point function that η is strictly positive for the stable fixed point. As a consequence, for $g>0$ correlations decay at large distance faster than in the Gaussian theory.

A more general example: Cubic anisotropy, a model with two couplings and four fixed points

One considers an N-component field ϕ_α, $\alpha = 1, \dots, N$ and a Hamiltonian invariant under the cubic group, the finite group of transformations generated by

$$\phi_\alpha \mapsto -\phi_\alpha\,, \quad \phi_\alpha \leftrightarrow \phi_\beta \quad \text{for all } \alpha \text{ and } \beta.$$

The cubic symmetry group admits a unique quadratic invariant, which ensures that all components of ϕ are critical simultaneously, but two independent quartic invariants of ϕ^4 type. The corresponding symmetric ϕ^4-like theory has the general form

$$\mathcal{H}(\phi)$$

$$= \int \mathrm{d}x \left\{ \tfrac{1}{2} \sum_\alpha \left[\big(\nabla \phi_\alpha(x)\big)^2 + r\phi_\alpha^2(x) + \frac{h}{4!}\phi_\alpha^4(x) \right] + \frac{g}{4!} \Big(\sum_\alpha \phi_\alpha^2(x) \Big)^2 \right\}.$$

RG and fixed points

The RG flow equations have the general form

$$\begin{aligned} \lambda\frac{\mathrm{d}g}{\mathrm{d}\lambda} &= -\beta_g\big(g(\lambda),h(\lambda)\big), \\ \lambda\frac{\mathrm{d}h}{\mathrm{d}\lambda} &= -\beta_h\big(g(\lambda),h(\lambda)\big). \end{aligned}$$

A simple calculation determines the two β-functions at leading order for $g,h=O(\varepsilon)$, $\varepsilon=4-d\to 0_+$:

$$\begin{aligned} \beta_g(g,h) &= -\varepsilon g+\frac{1}{8\pi^2}\left(\frac{N+8}{6}g^2+gh\right), \\ \beta_h(g,h) &= -\varepsilon h+\frac{1}{8\pi^2}\left(2gh+\frac{3}{2}h^2\right). \end{aligned}$$

Fixed points. One finds:

(i) The Gaussian fixed point: $g = h = 0$.

(ii) The decoupled fixed point: $g = 0\,, \quad h = 16\varepsilon\pi^2/3$,

which corresponds to N identical and decoupled copies of an Ising type model with a $\mathbb{Z}_2$ reflection symmetry.

(iii) The isotropic fixed point (which is always present): $h = 0\,, \quad g = 48\varepsilon\pi^2/(N+8)$,

which has an $O(N)$ symmetry, more extended than the cubic symmetry of the initial Hamiltonian.

(iv) Finally, a new fixed point, the cubic fixed point:

$$g = \frac{16\pi^2\varepsilon}{N}\,, \qquad h = \frac{16\pi^2(N-4)\varepsilon}{3N}\,.$$

Linearized flow and eigenvalues

The local stability of the four fixed points is determined by the eigenvalues of the matrix of partial derivatives, with respect to g and h, of the functions $-\beta_g, -\beta_h$, positive eigenvalue meaning instability:

$$\begin{aligned}
\text{Gaussian fixed point:}\quad & \varepsilon\,, && \varepsilon\,,\\
\text{Decoupled (Ising) fixed point:}\quad & \tfrac{1}{3}\varepsilon\,, && -\varepsilon\,,\\
\text{Isotropic fixed point:}\quad & \frac{N-4}{N+8}\varepsilon\,, && -\varepsilon\,,\\
\text{Cubic fixed point:}\quad & \frac{4-N}{3N}\varepsilon\,, && -\varepsilon\,.
\end{aligned}$$

Both the Gaussian and decoupled fixed points are always unstable.

The isotropic fixed point is stable for $N < N_c$ with $N_c = 4 + O(\varepsilon)$, a special example of a general result. Finally, the cubic fixed point is stable for $N > N_c$. At $N = N_c$ the two fixed points merge and then exchange role.

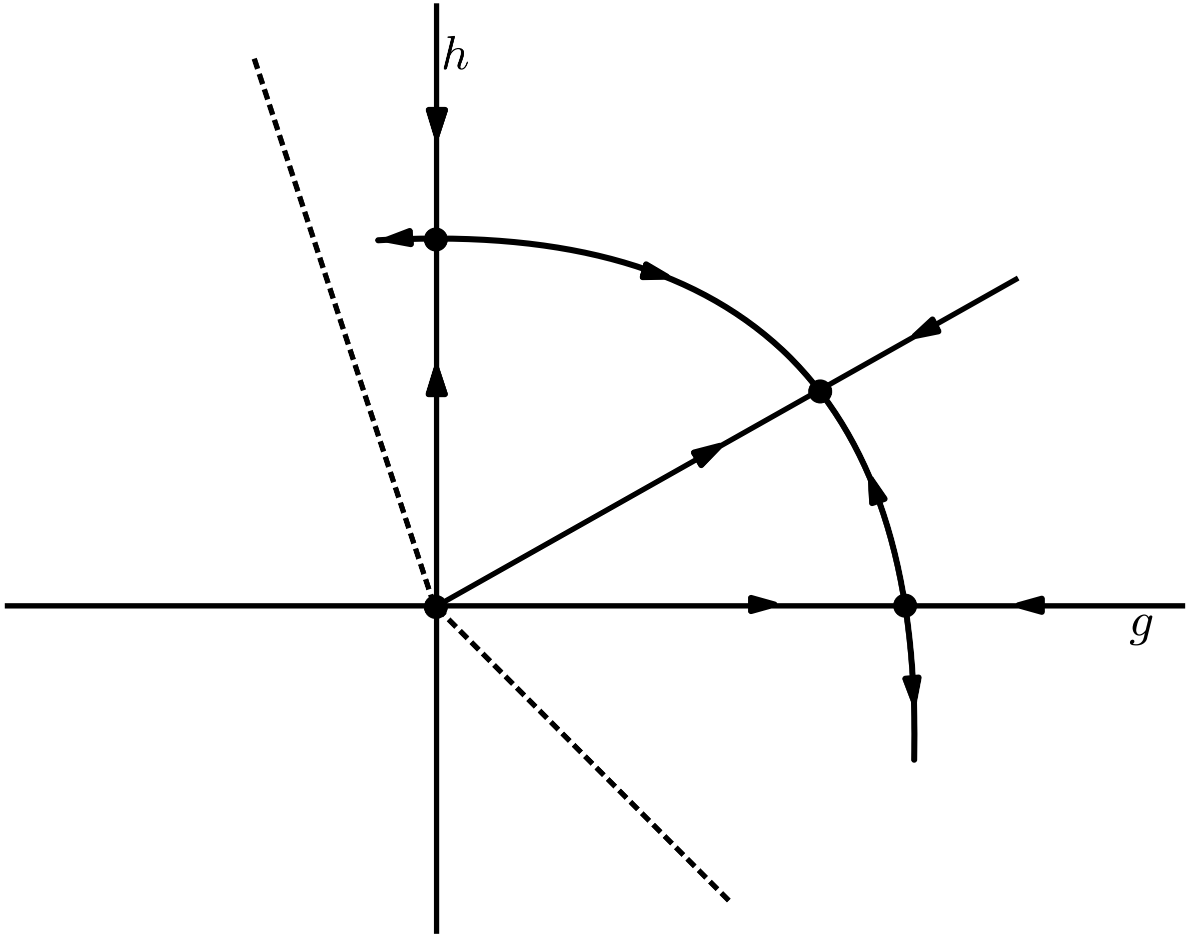

Fig. 1 Cubic anisotropy: RG flow for $N > 4$.

Corresponding values of the exponent η

The corresponding values of the exponent η are

$$\begin{aligned}
\text{Gaussian fixed point:}\quad & \eta = 0\,, \\
\text{Decoupled (Ising) fixed point:}\quad & \eta = \frac{\varepsilon^2}{54}\,, \\
\text{Isotropic fixed point:}\quad & \eta = \frac{(N+2)\varepsilon^2}{2(N+8)^2}\,, \\
\text{Cubic fixed point:}\quad & \eta = \frac{(N+2)(N-1)\varepsilon^2}{54N^2}\,.
\end{aligned}$$

One verifies that the largest value of η always corresponds to the stable fixed point.

Brézin, Le Guillou and Zinn-Justin generalized this property to a generic situation with four fixed points (see the review in the Domb and Green series (1976).

General Hamiltonian with N-component field: RG analysis

We consider only Hamiltonians that are invariant under a symmetry group G such the quadratic invariant in the field is unique. The two-point function in the disordered phase thus has the form:

$$\langle \phi_i(x)\phi_j(y)\rangle \equiv W^{(2)}_{ij}(x-y) = \delta_{ij} W^{(2)}(x-y).$$

This ensures that all components of the order parameter become critical simultaneously.

A general Hamiltonian satisfying the assumption can be written as

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left\{ \frac{1}{2} \sum_{i=1}^{N} \left[(\nabla \phi_i)^2 + (r_c + t)\,\phi_i^2 \right] + \frac{1}{4!} \sum_{i,j,k,l=1}^{N} g_{ijkl}\phi_i\phi_j\phi_k\phi_l \right\},$$

where g_{ijkl} is a tensor symmetric in its four indices. As a consequence of the symmetry group G, the tensor g_{ijkl} has special properties that take the form of successive constraints in the perturbative expansion.

RG functions as ε-expansions.

The RG β-functions, in the minimal subtraction scheme, up to order ε^3 (and g assumed of order ε) are given by

$$\begin{aligned}\beta_{ijkl}(g) = -\varepsilon g_{ijkl} + \frac{N_d}{2}\sum_{m,n}\left(g_{ijmn}g_{mnkl} + g_{ikmn}g_{mnjl} + g_{ilmn}g_{mnkj}\right). \\ - \frac{N_d^2}{4}\sum_{m,n,p,q}\left(g_{ijmn}g_{mpqk}g_{npql} + 5 \text{ terms}\right) \\ + \frac{N_d^2}{48}\sum_{m,n,p,q}\left(g_{ijkm}g_{mnpq}g_{npql} + 3 \text{ terms}\right) + O\left(g^4\right),\end{aligned}$$

where $N_d = 2/(4\pi)^{d/2}\Gamma(d/2)$ is the usual loop factor.

At order g^3, the field dimension can be inferred from the function

$$\eta(g) = \frac{N_d^2}{24N} \sum_{i,j,k,l} g_{ijkl} g_{ijkl} - \frac{N_d^3}{32N} \sum_{i,j,k,l,m,n} g_{ijkl} g_{klmn} g_{mnij} + O(g^4).$$

The result is consistent with the general result $\eta(g) \geq 0$.

In the latter expression, the diagonality of the two-point function has been used explicitly, which implies

$$\sum_{k,l,m} g_{iklm} g_{jklm} = \frac{\delta_{ij}}{N} \sum_{k,l,m,n} g_{klmn} g_{klmn} \,,$$

$$\sum_{k,l,mn,p} g_{iklm} g_{lmnp} g_{npkj} = \frac{\delta_{ij}}{N} \sum_{i,j,k,l,m,n} g_{qklm} g_{lmnp} g_{npkq} \,.$$

Gradient property

One verifies from the general expression that the β-function, at this order, derives from a potential (Wallace and Zia). Indeed,

$$\beta_{ijkl}(g) = \frac{\partial U(g)}{\partial g_{ijkl}}$$

with

$$\begin{aligned} U(g) = -\frac{\varepsilon}{2} \sum_{i,j,k,l} g_{ijkl} g_{ijkl} + \frac{N_d}{2} \sum_{i,j,k,l,m,n} g_{ijkl} g_{klmn} g_{mnij} \\ - \frac{3N_d^2}{8} \sum_{i,j,k,l,m,n,p,q} g_{ijmn} g_{mpqk} g_{npql} g_{ijkl} \\ + \frac{N_d^2}{48} \sum_{i,j,kl,m,n,p,q} g_{ijkl} g_{ijkm} g_{mnpq} g_{npql} + O(g^5)\,. \end{aligned}$$

General gradient flow

More generally, it has been verified up to order g^5 (i.e., all known orders) that the β-functions of the general ϕ^4 models can be written as ($g_a \equiv \{g_{ijkl}\}$)

$$\beta_a(g) = \sum_b T_{ab}(g) \frac{\partial U}{\partial g_b},$$

where the matrix $\mathbf{T}$ with elements T_{ab} is a symmetric positive matrix, regular function of the g_a, at least in the vicinity of $g_a = 0$.

The equation

$$\lambda \frac{\mathrm{d} g_a}{\mathrm{d}\lambda} = -\beta_a\bigl(g(\lambda)\bigr),$$

then defines a gradient flow.

Note that this general form of the β-function is the only one consistent with the transformation properties under reparametrization in the space of the coefficients g_a (diffeomorphisms).

Consequences

The property of gradient flow has several consequences:

(i) Because T_{ab} is a positive matrix, fixed points are extrema of the potential:

$$\beta_a(g^*) = 0 \;\Leftrightarrow\; \frac{\partial U(g^*)}{\partial g_a} = 0\,.$$

Moreover, the potential decreases along an RG trajectory.

(ii) The eigenvalues of the matrix of first order partial derivatives of β at a fixed point are real.

(iii) Stable fixed points are local minima of the potential, that is, the matrix of second derivatives

$$\frac{\partial^2 U(g^*)}{\partial g_a \partial g_b}$$

is positive.

Fixed points stability and value of the potential

In the framework of the ε-expansion, we prove two consequences of the property of gradient flow: there exists at most one stable fixed point; the stable fixed point corresponds to the lowest value of the potential.

Indeed, let us assume the existence of two fixed points corresponding to the parameters g^* and g'^*. We then consider the parameters g of the form

$$g(s) = sg^* + (1-s)g'^*\,, \quad 0 \le s \le 1\,,$$

and the corresponding potential

$$u(s) = U\bigl(g(s)\bigr).$$

As the explicit form shows, at leading order $u(s)$ is a third degree polynomial in s.

Due to the fixed point conditions at $s = 0$ and $s = 1$, its derivative

$$u'(s) = \sum_a g'_a(s) \frac{\partial U}{\partial g_a} = \sum_a (g^*_a - g'_a{}^*) \frac{\partial U}{\partial g_a}$$

vanishes at $s = 0$ and $s = 1$: $u'(0) = u'(1) = 0$. Since $u'(s)$ is a second degree polynomial, it then has necessarily the form

$$u'(s) = As(1 - s).$$

The second derivative $u''(s)$ is given in terms of the matrix of second partial derivatives of U and, thus, the partial derivatives of the β-functions, by

$$u''(s) = \sum_{a,b} (g^*_a - g'_a{}^*) \frac{\partial^2 U\big(g(s)\big)}{\partial g_a \partial g_b} (g^*_b - g'_b{}^*) = A(1 - 2s)\,.$$

In particular, for $s=0$ and $s=1$,

$$A=\sum_{a,b}(g_a^*-g_a'^{\,*})\frac{\partial^2 U(g'^*)}{\partial g_a \partial g_b}(g_b^*-g_b'^{\,*})\,,\ -A=\sum_{a,b}(g_a^*-g_a'^{\,*})\frac{\partial^2 U(g^*)}{\partial g_a \partial g_b}(g_b^*-g_b'^{\,*})\,.$$

At a stable fixed point, the matrix $\mathbf{U}''$ of partial second derivatives of U is positive. Thus, if g^* and g'^* are stable fixed points, A and $-A$ are both given by the expectation value of a positive matrix and thus are both positive, which is contradictory: both fixed points cannot be stable.

More generally, the sign of A characterizes, in some sense, the relative stability of these two fixed points. Let us assume, for example, $A<0$ which is consistent with the assumption that g^* is stable. Then $u'(s)<0$ in $[0,1]$ and $U(g(s))$ is a decreasing function. Thus,

$$U(g^*)<U(g'^*).$$

In particular, if g^* is a stable fixed point, it corresponds, among all fixed points, to the lowest value of the potential.

Fixed point stability and field dimension

For any fixed point g^*, the equation $\beta = 0$ implies, at leading order

$$\left.\frac{\mathrm{d}U(\lambda g^*)}{\mathrm{d}\lambda}\right|_{\lambda=1} = 0 \;\Rightarrow\; \varepsilon \sum_{i,j,k,l} g^*_{ijkl} g^*_{ijkl} = \frac{3N_d}{2} \sum_{i,j,k,l,m,n} g^*_{ijkl} g^*_{klmn} g^*_{mnij}$$

and, thus,

$$U(g^*) = -\tfrac{1}{6}\varepsilon \sum_{i,j,k,l} g^*_{ijkl} g^*_{ijkl} + O(g^4),$$

a negative value and thus lower than the Gaussian fixed point value

$$g^* \neq 0 \;\Rightarrow\; U(g^*) < U(0).$$

A relation between value of η and fixed point stability then follows from

$$\eta = \frac{N_d^2}{24N} \sum_{i,j,k,l} g^*_{ijkl} g^*_{ijkl} = -\frac{N_d^2}{4N\varepsilon} U(g^*).$$

The stable fixed point corresponds to the lowest value of U.

It thus corresponds also to the largest value of the exponent η and, as a consequence, of the dimension d_ϕ of the field ϕ:

$$d_\phi = \tfrac{1}{2}(d-2+\eta).$$

Since

$$\langle\phi(x)\phi(0)\rangle \underset{|x|\to\infty}{\propto} 1/|x|^{2d_\phi},$$

the correlation functions corresponding to the stable fixed point have the fastest large distance decay.

Beyond the ε-expansion

Vicari and Zinn-Justin have investigated the results known beyond the ε-expansion. Available numerical and exact results in three and two dimensions, to the best of our knowledge, are all consistent with the conjecture provided one adds the condition that the fixed points that are compared can be related by RG flows.

Examples include:

The $O(M) \times O(N)$ model in the large N limit. Some $O(M) \times O(N)$ models have also been investigated in 3D directly by numerical techniques for finite values of M and N. In particular, much is known for $M = 2$ and small values of N from resummed perturbation theory.

Models with $O(N)$ symmetry and cubic anisotropy.

$U(N) \times U(N)$ symmetric models.

The $SU(4)$ model.

The spin-density-wave model, which depends on five parameters.

Conclusion

We have proven within the framework of the ε-expansion, and verified by various techniques at fixed dimension, that the RG stable IR or long distance fixed point corresponds to the largest value of the exponent η and, as a consequence, of the dimension d_ϕ of the field ϕ: therefore, the correlation functions corresponding to the stable fixed point have the fastest large distance decay.

The general validity of this result, beyond the ε-expansion, remains a conjecture, even if it seems to be quite reasonable.

Possible extensions

Other verifications

It is clear that our work does not exhaust all possible verifications. Even within the ε-expansion, one could try to push the calculations to next order in the situation where two fixed points merge.

More general models

Other field theories could be explored, including, for example, fermions and gauge fields.

Multicritical points

In systems where several independent quadratic invariants in the fields can be found, the condition that all field components are critical simultaneously requires fixing new parameters in addition to the temperature and thus correspond to multicritical points. The unique exponent η is then replaced by a matrix. The conjecture can then be extended in the following form:

The stable fixed point corresponds to the η-matrix with the largest trace.

This conjecture has been less explored that the previous one. Still one case has been studied beyond the ε-expansion, a model with $O(n_1)\oplus O(n_2)$ symmetry, with two fields ϕ_1 and ϕ_2 with n_1 and n_2 components, respectively. The Hamiltonian reads

$$\begin{aligned}\mathcal{H} = {} & \tfrac{1}{2}(\partial_\mu\phi_1)^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 + \tfrac{1}{2}r_1\phi_1^2 + \tfrac{1}{2}r_2\phi_2^2 + \tfrac{1}{4!}g_1\left(\phi_1^2\right)^2 + \tfrac{1}{4!}g_2\left(\phi_2^2\right)^2 \\ & + \tfrac{1}{12}g_{12}\phi_1^2\phi_2^2 .\end{aligned}$$