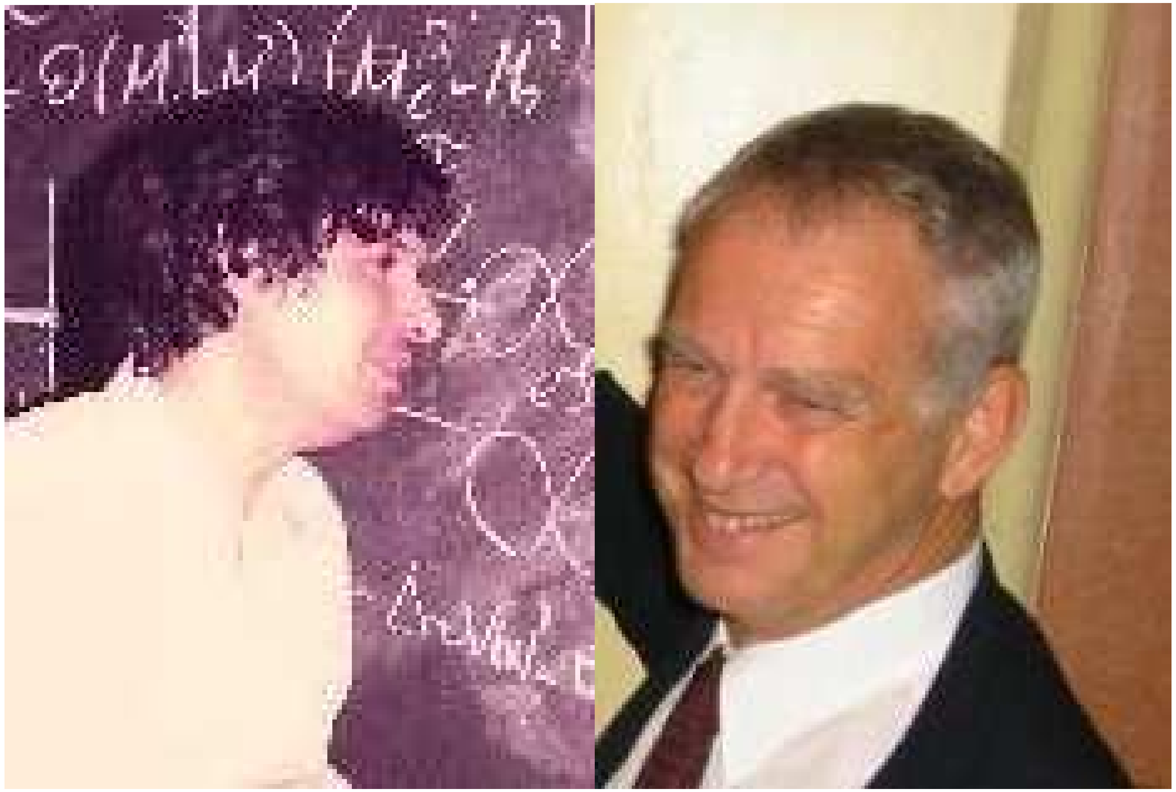
Revisiting old matrix integrals

J.-B. Zuber

To Édouard and Giorgio

as a testimony of admiration and friendship



Matrix integrals

$$\mathcal{Z} = \int_G d\Omega \exp N\beta \Re e\,(\mathrm{tr}\,(\Omega J))$$

or

$$Z = \int_G d\Omega \exp N\beta \Re e\,(\mathrm{tr}\,(A\Omega B\Omega^\dagger))$$

over a compact group G, are frequently encountered in physics (and in maths) : “Bessel matrix functions”. They have been mostly studied for $G = \mathrm{U}(N)$ ($\beta = 2$).

What happens when $G = \mathrm{O}(N)$ ($\beta = 1$)?

If A and B are both real skew-symmetric (i.e. in the Lie algebra of $G = \mathrm{O}(N)$), Z is known exactly from the work of Harish-Chandra ’57.

If A and B are both real symmetric, much more complicated and elusive, see [Brézin and Hikami ’02-06].

Expect things to simplify as $N \longrightarrow \infty$ [Weingarten '78]. Look at the "free energies" :

$$W(J.J^\dagger) = \lim_{N \longrightarrow \infty} \frac{1}{N^2} \log \mathcal{Z}$$

and

$$F(A,B) = \lim_{N \longrightarrow \infty} \frac{1}{N^2} \log Z$$

Then $W(X)$ and $F(A,B)$ are, up to an overall factor, independent of $G = \mathrm{O}(N),\ \mathrm{U}(N)$!

(Not true at finite N !)

More precisely,

$$W_{\mathrm{O}}(J.J^{\dagger}) = \frac{1}{2} W_{\mathrm{U}}(J.J^{\dagger}) \tag{1}$$

and

$$F_{\mathrm{O}}(A,B) = \frac{1}{2} F_{\mathrm{U}}(A,B) \tag{2}$$

Proof relies either on inspection of explicit formulae (skew-symmetric case), or on the use of differential equations satisfied by $\mathcal{Z}$, resp. Z, which simplify in the $N \longrightarrow \infty$ limit.

For $\mathcal{Z}_{\rm O} = \int_{{\rm O}(N)} dO \exp N \mathrm{tr}\,(J.O)$, follow the steps of [Brézin-Gross '80]: the trivial identity $\sum_j \frac{\partial^2 \mathcal{Z}_{\rm O}}{\partial J_{ij} \partial J_{kj}} = N^2 \delta_{ik} \mathcal{Z}_{\rm O}$ is reexpressed in terms of the eigenvalues λ_i of the real symmetric matrix $J.J^t$:

$$4\lambda_i \frac{\partial^2 \mathcal{Z}_{\rm O}}{\partial \lambda_i^2} + \sum_{j \neq i} \frac{2\lambda_j}{\lambda_j - \lambda_i} \left(\frac{\partial \mathcal{Z}_{\rm O}}{\partial \lambda_j} - \frac{\partial \mathcal{Z}_{\rm O}}{\partial \lambda_i} \right) + 2N \frac{\partial \mathcal{Z}_{\rm O}}{\partial \lambda_i} = N^2 \mathcal{Z}_{\rm O} \; .$$

Writing as above $\mathcal{Z}_{\rm O} = \mathrm{e}^{N^2 W_{\rm O}}$ and dropping subdominant terms in the large N limit, with $W_{\rm O}$ and $W_i := N \partial \mathcal{W}_{\rm O} / \partial \lambda_i$ of order 1, we get

$$4\lambda_i W_i^2 + 2W_i + \frac{1}{N} \sum_{j \neq i} \frac{2\lambda_i}{\lambda_j - \lambda_i} (W_j - W_i) = 0 \qquad (3)$$

which is precisely the equation satisfied by $\frac{1}{2} W_{\rm U}$ in [B-G]. This suffices to complete the proof of (1).

An explicit expression of F_{U} is known [O' Brien-Z '84]

$$W_{\mathrm{U}}(J.J^{\dagger}) = \sum_{n=1}^{\infty} \sum_{\alpha \vdash n} W_{\alpha} \frac{\mathrm{tr}_{\alpha} J.J^{\dagger}}{\prod_p (\alpha_p!\, p^{\alpha_p})}$$

$$W_{\alpha} = (-1)^n \frac{(2n + \sum \alpha_p - 3)!}{(2n)!} \prod_{p=1}^{n} \left(\frac{-(2p)!}{p!(p-1)!} \right)^{\alpha_p} ,$$

where $\alpha \vdash n$ denotes a partition of $n = \alpha_1.1 + \alpha_2.2 + \cdots + \alpha_n.n$ and

$$\mathrm{tr}_{\alpha}(X) := \prod_{p=1}^{n} \frac{1}{N} (\mathrm{tr}\, X^p)^{\alpha_p} .$$

For $Z^{(\mathrm{O})} = \int_{\mathrm{O}(N)} dO \exp N \mathrm{tr}\left(AOBO^t\right)$, take A and B both skew-symmetric, or both symmetric.

• A and B both skew-symmetric [Harish-Chandra]

block-diagonal form $A = \mathrm{diag}\left(\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}_{i=1,\cdots,m}\right)$, B likewise

$$Z^{(\mathrm{O})} = \mathrm{const.} \frac{\det(2\cosh 2Na_ib_j)}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}$$

(for O($N = 2m$)), with $\Delta_{\mathrm{O}}(a) = \prod_{1\leq i<j\leq m}(a_i^2 - a_j^2)$.

Regard A as $N \times N$ anti-Hermitian, eigenvalues $A_j = \pm i a_j$, B likewise. Easy to check that as $N \to \infty$,

$$Z^{(\mathrm{U})}(A,B) = \frac{\det\left(\mathrm{e}^{2NA_iB_j}\right)}{\Delta(a)\Delta(b)} \sim \left(\frac{(\det(\mathrm{e}^{2Na_ib_j})_{1\leq i,j\leq m}}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}\right)^2 = (Z^{(\mathrm{O})(A,B)})^2$$

- A and B both symmetric

Can take them in diagonal form $A = \operatorname{diag} a_i$, $B = \operatorname{diag} b_i$

Can prove that in the large N limit,

$$\sum_i \left(\frac{N}{\beta} \frac{\partial F^{(G)}}{\partial a_i} + \frac{1}{2N} \sum_{j \neq i} \frac{1}{a_i - a_j} \right)^p = \operatorname{tr} B^p$$

Hence $F^{(\mathrm{O})}$ ($\beta = 1$) satisfies same set of equations as $F^{(\mathrm{U})}$ ($\beta = 2$), q.e.d.

Where does this equation come from ? Define the differential operator $D_p(\partial/\partial A)$ by $D_p(\partial/\partial A)\mathrm{e}^{N\mathrm{tr}\,AB} = N^p\mathrm{tr}\,B^p\,\mathrm{e}^{N\mathrm{tr}\,AB}$

If D_p acts on an invariant function $F(A) = F(\Omega A\Omega^\dagger)$, how to write it in terms of $\partial/\partial a_i$? For $G = \mathrm{U}(N)$,

$$D_p(\partial/\partial A) = \mathrm{tr}\left(\frac{\partial}{\partial A}\right)^p := \sum_{i_1,\cdots i_p}\frac{\partial}{\partial A_{i_1 i_2}}\frac{\partial}{\partial A_{i_2 i_3}}\cdots\frac{\partial}{\partial A_{i_p i_1}}$$

and

$$D_p = \frac{1}{\Delta(a)}\sum_i\left(\frac{\partial}{\partial a_i}\right)^p\Delta(a)\ .$$

$\Delta(a) = \prod_{i<j}(a_i - a_j)$ (a non trivial calculation !) [Itzykson–Z '80].

"Radial" expression of D_p given [Bergère and Eynard '08] by

$D_p = \sum_{i,j}(K^p)_{ij}$ with K a *matrix* differential operator

$K_{ii} = \frac{\partial}{\partial a_i} + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{a_i - a_j}$ and for $i \neq j$, $K_{ij} = -\frac{\beta}{2}\frac{1}{a_i - a_j}$.

Another proof [Guionnet-Zeitouni '02] (for A and B symmetric) as a by-product of the construction of F_G as the unique solution of Matytsin '94 flow: β dependence is explicit.

A Conjecture $F^{(\mathrm{O})}(A,B) = \frac{1}{2}F^{(\mathrm{U})}(A,B)$ extends to A and B generic (neither symmetric, nor skew-symmetric). Some evidence from power expansion.

Origin of this universality? Diagrammatics ? Relation between $\mathcal{Z} = \int_G d\Omega \exp N\beta\,\Re e\,(\mathrm{tr}\,(\Omega J))$ and $Z = \int_G d\Omega \exp N\beta\,\Re e\,(\mathrm{tr}\,(A\Omega B\Omega^\dagger))$?

In the case of $\mathrm{U}(N)$, yes [P. Zinn-Justin–Zuber '03], [Collins '03].
For $\mathrm{O}(N)$??

Particular case where A is of finite *rank*. Then in the expansion of $F = \sum \prod (\frac{1}{N} \mathrm{tr} A^{p_i}) \prod (\frac{1}{N} \mathrm{tr} B^{q_j})$, terms with a single trace of A dominate.

In the $\mathrm{U}(N)$ case (and $N \to \infty$) ([IZ '80])

$$F^{(\mathrm{U})} \sim \sum_{p=1}^{\infty} \frac{1}{p} (\frac{1}{N} \mathrm{tr} A^p) \psi_b(B)$$

where $\psi_p(B) = p$-th "non-crossing cumulant" of B ([BIPZ '78]).

Application : ⋆ The β universality of F_G first pointed out in that finite rank case [Marinari, Parisi, Ritort, '94], Spin glass Hamiltonian with n replicas of N Ising spins

$$\mathcal{H} = \sum_{i,j=1}^{N} \underbrace{\sum_{a=1}^{n} \sigma_i^a \sigma_j^a}_{\Omega_{ij}} O_{ij} \qquad \Omega \text{ of rank } \leq n$$

with a coupling O_{ij}, a real, orthogonal, symmetric matrix with an equal number of ± 1 eigenvalues, $O = V^t.D.V$.

Have to compute $Z = \int_{\mathrm{O}(N)} dV \exp \beta \operatorname{tr} DV\Omega V^t$.

Now according to Marinari, Parisi, Ritort, pretend you integrate over the unitary group,
compute $\sum \frac{1}{p} \operatorname{tr} \Omega^p \psi_p(D) =: \operatorname{tr} G(\Omega)$

and . . .

in the most general case.

For $\nu = \frac{1}{2}$, by inverting the second relation, after some algebra we find (we omit the suffix $\nu = \frac{1}{2}$ for G and ψ)

$$G(z) = \int_0^1 dt \, \frac{\sqrt{1 + 4z^2 t^2} - 1}{2z} \tag{30}$$

which gives

$$G'(z) = \frac{\psi(z) - 1}{z}. \tag{31}$$

After integrating the last relation with the condition $G(0) = 0$ we find

$$G(z) = \tfrac{1}{2} \log(\sqrt{1 + 4z^2} - 1) - \tfrac{1}{2} \log(2z^2) + \tfrac{1}{2}\sqrt{1 + 4z^2} - \tfrac{1}{2} \tag{32}$$

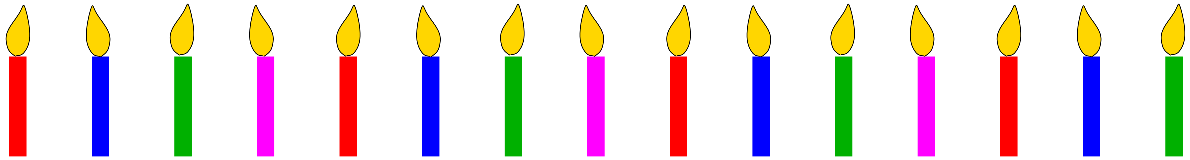
where the constant term has be chosen such that $G(0) = 0$.

We have already said that we have obtained this G for V unitary. It is easy to argue that when we integrate over orthogonal matrices the only difference is that $G(\beta z)$ gets substituted from $\frac{1}{2}G(2\beta z)$. This can be seen, for example, by noticing that the function G has to be the same in the two cases (since the same diagrams contribute) and at first order in β orthogonal and unitary matrices have to give the same results. So the only allowed renormalization will be of the kind $G(z) \to \alpha G(z/\alpha)$. The counting of the eigenvalues leads to the conclusion $\alpha = \frac{1}{2}$.

Using the fact that for integer positive k

!!??

which shows, once again, Giorgio's and his friends' great insight !



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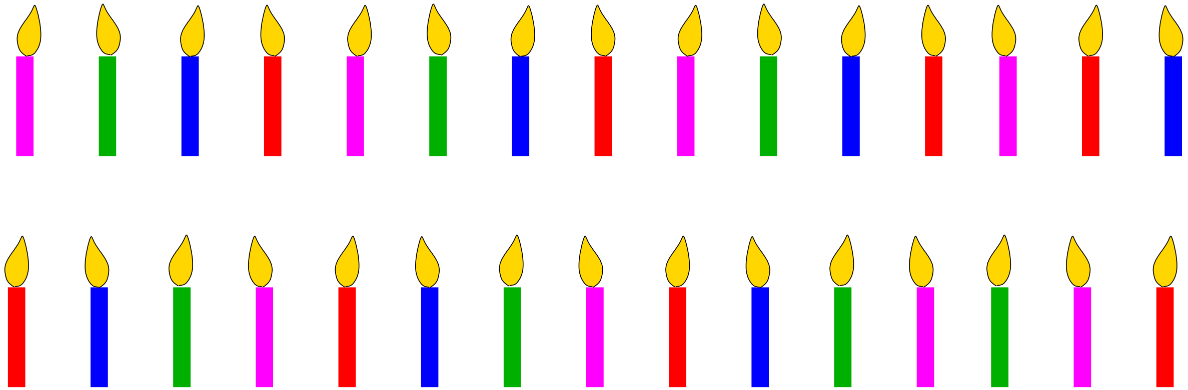

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