# Change of Variables Formula in Multiple Integrals

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## 1. The statement of the change of variables formula

Multiple (Riemann) integrals can be defined in a similar fashion as the Riemann integrals in the one variable case. Instead of defining integrals only on closed intervals in the one variable case, there is often need to define integrals over more general sets in multi-dimensions. However, complications arise in extending to multi-dimensions the concept of partitions over general sets; there are even bounded open sets $U$ in $\mathbb{R}^n$ for which $\int_U f(x)dx$ is not well defined in this fashion for $f$ that are continuous over the closure $\bar{U}$ of $U$.

However, it is not too hard to establish the following two facts:

1. If $f(x)$ is continuous and has compact support in $\mathbb{R}^n$, then the Riemann integral $\int_{\mathbb{R}^n} f(x)dx$ is well defined.

2. Suppose $U$ is a bounded open set in $\mathbb{R}^n$ such that its boundary $\partial U$ has a finite cover $\cup V_j$, and each $V_j \cap \partial U = \phi_j(W_j)$, where $W_j$ is a bounded open set in $\mathbb{R}^k$ for some $k \leq n-1$, and $\phi_j$ is Lipschitz over the closure $\bar{W}_j$ of $W_j$. Then for any $f(x)$ which is continuous over the closure $\bar{U}$ of $U$, the Riemann integral $\int_U f(x)dx$ is well defined.

**Remark.** *Although the cases covered above contain most cases that we normally encounter, Riemann integral has the obvious defect that it is not well defined for a wide enough class of functions and sets; in particular, reasonable limits of integrable functions may not be integrable. This lack of* *completeness of integrable functions is a major drawback of Riemann integrals: we would rather keep the completeness, but accept that integrals do not have to be defined in the Riemann fashion.*

*In this lecture, however, we do not assume that students have been exposed to Lebesgue's integration theory, and will state and discuss multiple integrals in the Riemann setting.*

Note that if $U$ is as in case 2 above, and $T : \bar{U} \mapsto \mathbb{R}^n$ is continuously differentiable and $1-1$ in $\bar{U}$, then $T(U)$ is also as in case 2.

**Theorem 1.** *Suppose $U$ is as in case 2 above, and $T : \bar{U} \mapsto \mathbb{R}^n$ is continuously differentiable and $1-1$ in $\bar{U}$. Then for* *any function* $f(y)$ *which is continuous over* $\bar{T}(U)$,

$$\int_{T(U)} f(y)dy = \int_U f(T(x))|J_T(x)|dx, \quad (1)$$

*where* $J_T(x) = \det\left(\frac{\partial T(x)}{\partial x}\right)$ *is the Jacobian of* $T$ *at* $x$.

**Remark.** *In the one-variable case, we don't put in the absolute value sign around the Jabobian, as our conventional notation encodes the orientation:*

$$\int_a^b \cdots dx = -\int_b^a \cdots dx.$$

**Example 1.** *One of the most commonly used change of variables is that from rectangular coordinates to polar coordinates:*

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}.$$

*The Jacobian of* $T$ *is* $J_T(r, \theta) = r$. *If* $U = \{(r, \theta) : 0 < r < R, 0 < \theta < 2\pi\}$, *then* $T$ *fails* *to be $1-1$ on a portion of $\partial U$ and $T(U)$ is not quite the open disc*

$$D_R = \{(x,y) : x^2+y^2 < R^2\}.$$

*However, for any $\epsilon > 0$ small, Theorem 1 is applicable on*

$$U_\epsilon = \{(r,\theta) : \epsilon < r < R, \epsilon < \theta < 2\pi\}.$$

*Thus for any function $f(x,y)$ which is continous over $\bar{D}_R$,*

$$\int_{T(U_\epsilon)} f(x,y)dxdy = \int_{U_\epsilon} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

*Then using*

$$\int_{D_R} f(x,y)dxdy = \lim_{\epsilon\to 0}\int_{T(U_\epsilon)} f(x,y)dxdy,$$

*and*

$$\int_U f(r\cos\theta, r\sin\theta)rdrd\theta$$
$$= \lim_{\epsilon\to 0}\int_{U_\epsilon} f(r\cos\theta, r\sin\theta)rdrd\theta,$$

*we obtain*

$$\int_{D_R} f(x,y)dxdy = \int_U f(r\cos\theta, r\sin\theta)rdrd\theta.$$

**Remark.** *Approximations as done above are often needed in situations where Theorem 1 can not be applied directly.*

*Just as in the one variable case, when $f$ is not necessarily continuous (or bounded), one can define improper integral. An examination of the limiting argument above shows that if $f(x,y)$ is known to be continuous away from the origin, and for some $C > 0$ and $1 > \delta > 0$, we have*

$$|f(x,y)| \le Cr^{-1-\delta},$$

*then the improper integral $\int_{D_R} f(x,y)dxdy$ is well defined and the change of variables formula above is still valid.*

**Example 2.** *Similarly, for spherical coordinate*

$$\begin{cases} x = & r\cos\theta\sin\phi, \\ y = & r\sin\theta\sin\phi, \\ z = & r\cos\phi, \end{cases}$$

*the Jacobian matrix is*

$$\begin{bmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{bmatrix}$$

*so the Jacobian is $r^2\sin\phi$. As an application, the volume of a ball of radius $R$ in $\mathbb{R}^3$ is*

$$\int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\phi drd\phi d\theta = \frac{4\pi R^3}{3}.$$

**Example 3.** *Suppose that $f(x,y)$ is radially symmetric in $\mathbb{R}^2$. We will prove that its Fourier transform*

$$F(\xi,\eta) = \int\int_{\mathbb{R}^2} f(x,y)\cdot e^{-i(x\cdot\xi+y\cdot\eta)}dxdy.$$

*is radially symmetric in $(\xi,\eta)$. This can be verified as follows. Given $(\xi,\eta)$. Let $R$ be the $2\times 2$ rotation matrix such that $(\xi,\eta)R = (\sqrt{\xi^2+\eta^2}, 0)$. Then introduce the transformation*

$$\begin{bmatrix} x \\ y \end{bmatrix} = R\begin{bmatrix} x' \\ y' \end{bmatrix}.$$

*Then* $x^2+y^2=(x')^2+(y')^2$ *and the Jacobian of this transformation is* 1. *So*

$$\begin{aligned} F(\xi,\eta) =& \int\int_{\mathbb{R}^2} f(x,y)\cdot e^{-i(x\cdot\xi+y\cdot\eta)}dxdy \\ =& \int\int_{\mathbb{R}^2} f(x',y')\cdot e^{-i\sqrt{\xi^2+\eta^2}x'}dx'dy' \\ =& F(\sqrt{\xi^2+\eta^2},0). \end{aligned}$$

## 2. Some discussion of proof

In the one variable case, the change of variable formula

$$\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx,$$

can be formally derived by substituting $dy = g'(x)dx$. However, in the multiple integral case, this formal procedure would not be valid. For instance, in the two dimensional case, suppose $x=\phi(u,v), y=\psi(u,v)$. Then

$dx = \phi_u du + \phi_v dv$, and $dy = \psi_u du + \psi_v dv$. So the formal substitution would produce

$$\begin{aligned} dxdy =& (\phi_u du + \phi_v dv)(\psi_u du + \psi_v dv) \\ =& \phi_u \psi_u (du)^2 + (\phi_u \psi_v + \phi_v \psi_u) dudv + \phi_v \psi_v (dv)^2, \end{aligned}$$

which not only does not produce the claimed substitution $dxdy = |\phi_u \psi_v - \phi_v \psi_u| dudv$, but also contain the terms $(du)^2$ and $(dv)^2$, with no obvious geometric meaning.

**Remark.** *If one treats $dxdy$ as the exterior product $dx \wedge dy$, then the exterior algebra rules give $dx \wedge dy = (\phi_u \psi_v - \phi_v \psi_u) du \wedge dv$. This is how exterior algebra enters into multi-variable calculus and geometry in a natural way.*

It turns out that the underlying geometric idea in Theorem 1 is embedeed in the case where we take $f \equiv 1$, and $T$ to be a linear transformation given by a matrix multiplication by a matrix $A$. Then Theorem 1 says that $Volume(A(U)) = |\det A| Volume(U)$.

This special case can be verified directly, first when $U$ is a rectangle, then by approximation of non-overlapping union of rectangles.

In the case $U$ is the standard unit cube, then $A(U)$ is a parallelopiped with the columns of $A$ as edges. First decompose $A = E_1 \cdots E_k$ as a product of elementary matrices, and define $F_j = F_{j-1}E_j$, with $F_0 = Id$, and let $P_j$ be the parallelopiped with the columns of $F_j$ as edges. Then $A(U) = P_k$. Because $\det A = \det E_k \cdots \det E_1$, it suffices to verify that

$$Volume(P_j) = |\det E_j| Volume(P_{j-1}), \text{for each } j.$$

Geometrically this is evident — the underlying principle being that $P_j$ is a parellelopiped transformed from the parellelopiped $P_{j-1}$ by one step of elementary column operation; the slightly non-trivial case corresponds to the case where $P_{j-1}$ and $P_j$ share all common edges with the exception of one, which, however, produce equal heights relative to the hyperplanes spanned by the common edges. Thus

$$Volume(A(U)) = |\det E_1| \cdots |\det E_k| Volume(U).$$

On the other hand, $\det A = \det E_1 \cdots \det E_k$. So $Volume(A(U)) = |\det A| Volume(U)$.

When $T$ is not necessarily a linear transformation, one first uses the ideas above to prove that

$$\int_{T(U)} f(y)dy \leq \int_U f(T(x)) \left| \det \left( \frac{\partial T(x)}{\partial x} \right) \right| dx, \tag{2}$$

for any continuous nonnegative function $f$, then apply the same to $T^{-1}$ on $T(U)$ with

$|\det\left(\frac{\partial T(x)}{\partial x}\right)|$ as the function to be integrated to obtain

$$
\begin{aligned}
&\int_{T(U)} dy \\
&\le \int_U \left|\det\left(\frac{\partial T(x)}{\partial x}\right)\right| dx \\
&\le \int_{T(U)} \left|\det\left(\frac{\partial T(x)}{\partial x}\right)\right|_{x=T^{-1}(y)} \left|\det\left(\frac{\partial T^{-1}(y)}{\partial y}\right)\right| dy \\
&\le \int_{T(U)} dy,
\end{aligned}
$$

using

$$\left|\det\left(\frac{\partial T(x)}{\partial x}\right)\right|_{x=T^{-1}(y)} \left|\det\left(\frac{\partial T^{-1}(y)}{\partial y}\right)\right| = 1.$$

Here the Inverse Function Theorem is needed to justify the discussion on $T^{-1}$.

(2) can be proved by Taylor's expansion of $T$:

$$T(x) = T(\bar{x}) + \left(\frac{\partial T(x)}{\partial x}\bigg|_{x=\bar{x}} + C(x,\bar{x})\right)(x-\bar{x})$$

where $|C(x, \bar{x})| \leq \epsilon$ as long as $|x - \bar{x}| \leq \delta$ for some $\delta > 0$ which depends on $T$ and $\epsilon > 0$. Let $Q$ be a cube with $\bar{x}$ as center and side length $h$. Then

$$\{T(\bar{x}) + \frac{\partial T(x)}{\partial x}\Big|_{x=\bar{x}} (x - \bar{x}) : x \in Q\}$$

is a parallelopiped with volume $|\det\left(\frac{\partial T(x)}{\partial x}{}_{x=\bar{x}}\right)|h^n$ as discussed above for the transformation of volume under a linear transformation. The Taylor expansion above means that $T(Q)$ is contained in a parallelopiped with $T(\bar{x})$ as center, with the same edges as those for the parallelopiped above but with side length $(1 + \epsilon)h$. Thus

$$Volume(T(Q)) \leq \left|\det\left(\frac{\partial T(x)}{\partial x}{}_{x=\bar{x}}\right)\right| (1+\epsilon)^n Volume(Q)$$

as long as the side length of $Q$ is no more than $\delta$. One then uses this estimate and partition to prove (2). For details, see *J.*

*Schwartz, The formula for change in variables in a multiple integral, Amer. Math. Monthly 61, (1954). 8185.*

## 3. Area formula and surface integral

An extension of the idea above can be used to define and compute the surface area and surface integral. Again we start with a discussion of a transformation defined by an $n \times m$ matrix $A$ with $m \leq n$.

$$T : x \in \mathbb{R}^m \mapsto Ax \in \mathbb{R}^n.$$

$T(\mathbb{R}^m)$ is an $m-$dimensional subspace of $\mathbb{R}^n$, and the standard cube $U$ in $\mathbb{R}^m$, $T(U)$ is a parallelopiped in $T(\mathbb{R}^m)$, with $col_j(A)$, $j = 1, \ldots, m$ as edges. Choose an orthonormal basis $\tau_1, \ldots, \tau_m$ of $T(\mathbb{R}^m)$, and express each $col_j(A)$ in terms of this basis:

$$col_j(A) = \sum_{i=1}^{m} B_{ij} \tau_i,$$

and let $B$ be the $m \times m$ matrix $(B_{ij})$. Then $T(U)$ can be thought of as obtained from the unit cube in $T(\mathbb{R}^m)$ by applying the linear transformation through multiplication by $B$. So by our discussion from last section,

$$Volume(T(U)) = |\det B|.$$

On ther other hand,

$$A = [\tau_1 \cdots \tau_m]B.$$

So

$$A^T A = B^T [\tau_1 \cdots \tau_m]^T [\tau_1 \cdots \tau_m] B = B^T B,$$

using

$$[\tau_1 \cdots \tau_m]^T [\tau_1 \cdots \tau_m] = I_m.$$

Now $\det B = \sqrt{\det B \det B^T} = \sqrt{\det(B^T B)} = \sqrt{\det(A^T A)}$, and we arrive at

$$Volume(T(U)) = \sqrt{\det(A^T A)} Volume(U).$$

Note that the $(i, j)$ entry $g_{ij}$ of $A^T A$, is $col_i(A) \cdot col_j(A) = Ae_i \cdot Ae_j$. This is one of the geometric origins for the Riemannian metric tensor and the appearance of the area (volume) form $\sqrt{\det(g_{ij})} dx$.

Based on the discussion above, suppose $U$ is a bounded open domain in $\mathbb{R}^m$ as in Theorem 1, and $F : \bar{U} \mapsto \mathbb{R}^n$, $n \geq m$, is differentiable and $1 - 1$ on $\bar{U}$, then

$$vol(F(U)) = \int_U \sqrt{\det(J_F^T(u) J_F(u))} du.$$

We can use Theorem 1 to check that this definition is independent of the parametrization, namely, if $\Phi : V \mapsto U$ is a diffeomorphism from $\bar{V}$ onto $\bar{U}$, then $\widehat{F}(v) = F \circ \Phi(v) : V \mapsto F(U)$ is another parametrization for $F(U)$, and we expect

$$\int_V \sqrt{\det(J_{\widehat{F}}^T(v) J_{\widehat{F}}(v)} dv = \int_U \sqrt{\det(J_F^T(u) J_F(u))} du.$$

This follows from Theorem 1 as follows.

$$\begin{aligned}&\int_U \sqrt{\det(J_F^T(u)J_F(u))}du\\ =&\int_V \sqrt{\det(J_F^T(\Phi(v))J_F(\Phi(v))}|\det J_\Phi(v)|dv.\end{aligned}$$

Noting that

$$J_{\widehat{F}}(v) = J_F(\Phi(v))J_\Phi(v),$$

so

$$J_{\widehat{F}}^T(v)J_{\widehat{F}}(v) = J_\Phi^T(v)J_F^T(\Phi(v))J_F(\Phi(v))J_\Phi(v).$$

Now $J_\Phi^T(v)$, $J_F^T(\Phi(v))J_F(\Phi(v))$, and $J_\Phi(v)$ are all square matrices, so

$$\begin{aligned}&\det(J_{\widehat{F}}^T(v)J_{\widehat{F}}(v))\\ =&\det(J_\Phi^T(v))\det(J_F^T(\Phi(v))J_F(\Phi(v)))\det(J_\Phi(v))\\ =&\det(J_F^T(\Phi(v))J_F(\Phi(v)))|\det(J_\Phi(v))|^2,\end{aligned}$$

resulting in

$$\begin{aligned}&\sqrt{\det(J_{\widehat{F}}^T(v)J_{\widehat{F}}(v))}\\ =&\sqrt{\det(J_F^T(\Phi(v))J_F(\Phi(v)))}|\det(J_\Phi(v))|.\end{aligned}$$

The computations above indicate that

$$\sqrt{\det(J_F^T(u)J_F(u))}du = \sqrt{\det(J_{\hat{F}}^T(v)J_{\hat{F}}(v))}dv$$

is a geometric quantity independent of the parametrization, and is called the volume (area) form of the submanifold (surface) $F(U)$, and denoted by $dVol$ or $dA$. The concept of tensors in geometry are introduced to deal with such quantities.

If $\rho(P)$ is a continuous function defined over $F(\bar{U})$, then the same computations show that $\int_{F(\bar{U})} \rho(P)dVol(P)$, which can be defined as $\int_U \rho(F(u))\sqrt{\det(J_F^T(u)J_F(u))}du$, is independent of the parametrization.

**Example 4.** *Let* $(u,v) \in U \mapsto x(u,v) \in \mathbb{R}^n$ *be a parametrization of a two dimensional surface in* $\mathbb{R}^n$*, and let*

$$\begin{aligned} E(u,v) &= x_u(u,v) \cdot x_u(u,v), \\ F(u,v) &= x_u(u,v) \cdot x_v(u,v), \\ G(u,v) &= x_v(u,v) \cdot x_v(u,v), \end{aligned}$$

*then*

$$Area(x(U)) = \int_U \sqrt{EG - F^2} dudv.$$

**Remark.** *The discussion above is based on one way of computing the area of a parallelogram with edges $\vec{u}$ and $\vec{v}$, namely, it is equal to $\sqrt{EG - F^2}$, with $E = \vec{u} \cdot \vec{u}$, $F = \vec{u} \cdot \vec{v}$, and $G = \vec{v} \cdot \vec{v}$. Another, more geometric approach, gives the area as $|\vec{u} \wedge \vec{v}|$. Thus we have the relation*

$$(\vec{u} \cdot \vec{u})\,(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 = |\vec{u} \wedge \vec{v}|^2,$$

*which can also be verified directly using algebra.*

**Example 5.** *Let $x \in U \subset \mathbb{R}^m \mapsto \mathbb{R}$ be a differentiable function over $\bar{U}$. Then its graph has a parametrization $x \in U \mapsto (x, f(x)) \in \mathbb{R}^{m+1}$, whose Jacobian matrix is*

$$\begin{bmatrix} I \\ \nabla f(x) \end{bmatrix}.$$

*So*

$$\begin{bmatrix} I \\ \nabla f(x) \end{bmatrix}^T \begin{bmatrix} I \\ \nabla f(x) \end{bmatrix}$$

$$= \begin{bmatrix} 1+f_1^2(x) & f_1(x)f_2(x) & \cdots & f_1(x)f_m(x) \\ f_2(x)f_1(x) & 1+f_2^2(x) & \cdots & f_2(x)f_m(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_m(x)f_1(x) & f_m(x)f_2(x) & \cdots & 1+f_m^2(x) \end{bmatrix},$$

*whose determinant can be computed to be* $1+|\nabla f(x)|^2$*. Therefore the area of the graph of* $f$ *over* $U$ *is*

$$\int_U \sqrt{1+|\nabla f(x)|^2}dx.$$

## 4. Green's and Stokes' Theorem

We will not attempt to give the most general form of these theorems or their proofs, but rather try to explain how the relevant concepts came about and how they are ofen applied.

In applications one often encounters the line integral of a vectorfield $\vec{F}$ along a path (or loop) $\Gamma$, which is defined as

$$\int_\Gamma \vec{F}\cdot d\vec{r} = \int_\Gamma \Big(F_1(r(t))r_1^{'}(t) + \cdots F_n(r(t))r_n^{'}(t)\Big)\, dt,$$

and the flux of $\vec{F}$ across a surface $S$, defined as

$$\int_S \vec{F}\cdot\vec{n}dA,$$

where $\vec{n}$ is a (continuous) choice of unit normal vector to $S$. One natural question is *what infinitesimal quantities determine the total line integral (called circulation) of $\vec{F}$ along a closed loop $\Gamma$ (or the total flux of $\vec{F}$ across a closed surface $S$)?*

One way to find such quantities is to examine the leading order term of $\int_\gamma \vec{F}\cdot d\vec{r}$ when $\gamma$ is a small loop around a spefic point $P$. For simplicity, let's assume $P$ to be the origin, and $\gamma$ is a planar loop contained in the plane spanned by a pair of orthonormal vectors $\vec{\xi}$ and $\vec{\eta}$:

$$\gamma(t) = x(t)\vec{\xi} + y(t)\vec{\eta}.$$

Assuming $\vec{F}$ to have the necessary differentiability, then Taylor expansion gives the leading order term of $\int_\gamma \vec{F} \cdot d\vec{r}$ to be

$$\sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} \int_\gamma \left( x(t)\xi_j + y(t)\eta_j \right) \left( x'(t)\xi_i + y'(t)\eta_i \right) dt$$

$$= \sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} \int_\gamma \left( x(t)x'(t)\xi_i\xi_j + y(t)y'(t)\eta_i\eta_j + x(t)y'(t)\xi$$

$$= \sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} \left( \xi_j\eta_i \int_\gamma x(t)y'(t)dt + \xi_i\eta_j \int_\gamma y(t)x'(t)dt \right)$$

using $\int_\gamma x(t)x'(t)dt = \int_\gamma y(t)y'(t)dt = 0$. But $\int_\gamma x(t)y'(t)dt = -\int_\gamma y(t)x'(t)dt =$ area enclosed by $\gamma$, so the leading order term of $\int_\gamma \vec{F} \cdot d\vec{r}$ is

$$\sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} \left( \xi_j\eta_i - \xi_i\eta_j \right) \cdot \text{(Area enclosed by } \gamma\text{)}$$

In the case of dimension 3, we can take

$\vec{n} = \vec{\xi} \wedge \vec{\eta} = (\xi_2\eta_3 - \xi_3\eta_2, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_1)$

to be the unit normal to the plane spanned by $\vec{\xi}$ and $\vec{\eta}$. Then

$$\sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} \left(\xi_j \eta_i - \xi_i \eta_j\right)$$

$$=\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) \cdot \vec{n},$$

thus producing the concept of the *curl* of a vector field.

In the general dimension, we see that

$$\sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} \left(\xi_j \eta_i - \xi_i \eta_j\right)$$

is a linear functional on oriented two planes $span\{\vec{\xi}, \vec{\eta}\}$, and is naturally identified to be the two form

$$\sum_{i,j} \frac{\partial F_i(P)}{\partial x_j} dx_j \wedge dx_i = \sum_i dF_i \wedge dx_i = d(\sum_i F_i dx_i).$$

In dimension 3, if a surface $S$ has a local parametrization $\vec{r}(u,v)$, and we take its unit normal to be $\vec{r}_u \wedge \vec{r}_v / |\vec{r}_u \wedge \vec{r}_v|$, then

$$\int_S \vec{F} \cdot \vec{n} dA$$
$$= \int_S \vec{F} \cdot \vec{r}_u \wedge \vec{r}_v du dv$$
$$= \int_S (F_1(y_u z_v - y_v z_u) + F_2(z_u x_v - x_u z_v) + F_3(x_u y_v - x_v$$

Note that $\int_S F_1(y_u z_v - y_v z_u) du dv$ is almost $\int_S F_1 dy dz$ by our change of variables formula. Exterior algebra is useful here to keep track of the signs, so using $dy \wedge dz = (y_u z_v - y_v z_u) du \wedge dv$, etc, we can rewrite $\int_S \vec{F} \cdot \vec{n} dA$ as

$$\int_S (F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) .$$

This formulation easily extends to general dimensions.

**Remark.** *Note that in order for integrals such as* $\int_S F_1 dy \wedge dz$ *to be independent of* *the parametrization used, the Jabobian between different parametrizations need to be positive, in other words, we need to be able to produce such "admissible" parametrizations, thus the notion of integrals of differential forms over "oriented" manifolds.*

The classical version of Green's Theorem is

**Theorem 2.** *Let $D$ be a domain in $\mathbb{R}^2$ whose boundary is a finite union of closed rectifiable curves, oriented positively relative to $D$, and $\vec{F} = (F_1, F_2)$ be a continuously differentiable vectorfield over $\bar{D}$. Then*

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \int_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

The classical version of Stoke's Theorem in dimension 3 is:

**Theorem 3.** *Let $S$ be an oriented surface in $\mathbb{R}^3$ whose boundary is a finite union of* *closed rectifiable curves, oriented positively relative to $D$, and $\vec{F} = (F_1, F_2, F_3)$ be a continuously differentiable vectorfield over $\bar{S}$. Then*

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_D \left( curl \vec{F} \cdot \vec{n} \right) dA.$$

The classical version of Divergence Theorem in dimension 3 is:

**Theorem 4.** *Let $D$ be a domain in $\mathbb{R}^3$ whose boundary is a finite union of $C^1$ surfaces, and $\vec{F} = (F_1, F_2, F_3)$ be a continuously differentiable vectorfield over $\bar{D}$. Then*

$$\int_{\partial D} \vec{F} \cdot \vec{n} dA = \int_D div \vec{F} dx dy dz,$$

*where $div \vec{F} = \frac{F_1}{\partial x} + \frac{F_2}{\partial y} + \frac{F_3}{\partial z}$ is the divergence of the vectorfield $\vec{F}$, and $\vec{n}$ is the unit exterior normal to $\partial D$.*

**Example 6.** *Compute the flux of the vector field $\vec{F} = \vec{r}/|\vec{r}|^2$ across a closed surface $S$ in $\mathbb{R}^3$ which encloses the origin.*

*Note that the vector field $\vec{F}$ is singular at the origin, and away from the origin, $div\vec{F} = 0$. Let $D$ denote the domain enclosed by $S$ and $D_1 = D \setminus B_\epsilon(0)$ for $\epsilon > 0$ small. Then the Divergence Theorem is applicable over $D_1$ with $\vec{F}$, and produces*

$$\int_{\partial D_1} \vec{F} \cdot \vec{n} dA = \int_{D_1} div\vec{F} dxdydz = 0.$$

*However,*

$$\int_{\partial D_1} \vec{F} \cdot \vec{n} dA = \int_S \vec{F} \cdot \vec{n} dA + \int_{\partial B_\epsilon(0)} \vec{F} \cdot \vec{n} dA,$$

*and over $\partial B_\epsilon(0)$, $\vec{n} = -\vec{r}/|\vec{r}|$, so*

$$\int_{\partial B_\epsilon(0)} \vec{F} \cdot \vec{n} dA = -\int_{\partial B_\epsilon(0)} \epsilon^{-2} dA = -4\pi.$$

*Thus we have proved that the flux of $\vec{F} = \vec{r}/|\vec{r}|^2$ across a closed surface $S$ in $\mathbb{R}^3$ which encloses the origin is $4\pi$. Note that for a surface that do not enclose the origin, a direct application of the Divergence Theorem gives its flux to be zero.*