The Classical Groups

IMR

Aug. 30, 2010

1 References

- Emil Artin, *Geometric Algebra*.
- J. Dieudonné, *La Géometrie des Groupes Classiques*. Packed with information, comprehensive.
- L. E. Dickson, *Linear Groups*. Uses older terminology
- D. E. Taylor, *The Geometry of the Classical Groups*. More recent, quite readable
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Fixed notation:
V is an n-dimensional vector space over a field k and $\mathcal{P}(V) = \mathcal{P}^{n-1}(k) =$ the corresponding projective space

The points of $\mathcal{P}(V)$ are the 1-dimensional subspaces of V.

$$\begin{array}{rcl} \Phi : V - \{0\} & \longrightarrow & \mathcal{P}(V) \\ v & \mapsto & kv \end{array}$$

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- The projective line: $\mathcal{P}^1(k) = k \cup \{\infty\}$ (think slopes)
- ♢ The projective plane: $\mathcal{P}^2(k) = k^2 \cup \ell_\infty$, $\ell_\infty \cong \mathcal{P}^1(k) =$ "line at infinity"
- $\mathcal{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} =$ "Riemann sphere"

3 Groups of symmetries of V and $\mathcal{P}(V)$

♢ ▶ $GL(V) := \{T : V \to V \mid T \text{ is an invertible lin. transf.}\}$
$\dots\ GL_n(k) = GL(n,k) := \{X \in Mat_{n\times n}(k) \mid X \text{ is invertible}\}$

$\forall$ ordered basis $B = \{v_1, \dots, v_n\}$ there is a group isomorphism
$GL(V) \cong GL_n(k),\ \ T \leftrightarrow [T]_B =$ matrix of T with resp. to B

$\det T := \det[T]_B$ is a good definition; for any other basis C,

♢ $[T]_C = M_{C,B}[T]_B M_{C,B}^{-1},\ M_{C,B} =$ "change of basis" matrix

▶ $SL(V) := \{T \in GL(V) \mid \det T = 1\}$
$\cong \quad SL_n(k) := \{X \in GL_n(k) \mid \det X = 1\}$

▶ $Z(V) := \{\gamma 1_V \mid \gamma \in k^\times\} \ \cong\ Z_n(k) := \{\gamma I \mid \gamma \in k^\times\} \ \cong\ k^\times$

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Lemma

Let $T \in GL(V)$. Then $T \in Z(V) \iff T$ fixes $\mathcal{P}(V)$ pointwise.

♢♢ ▸ $PGL(V) := GL(V)/Z(V)$ acts faithfully on $\mathcal{P}(V)$
$PGL_n(k) := GL_n(k)/Z_n(k)$, etc.

▸ $PSL(V) = SL(V)Z(V)/Z(V) \cong SL(V)/Z(V) \cap SL(V)$,
$PSL(V) \trianglelefteq PGL(V)$.

4 Bigger groups: $\Gamma L(V)$ and $P\Gamma L(V)$

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Definition

$S : V \to V$ *is a* ***semilinear transformation*** $\iff \exists \sigma \in Aut(k)$

$$S(\alpha v + \beta w) = \sigma(\alpha)S(v) + \sigma(\beta)S(w), \quad v, w \in V, \quad \alpha, \beta \in k.$$

$$\Gamma L(V) = \{S : V \to V \mid S \text{ is invertible semilin. transf.}\}$$

Example ("'Hermitian transpose")

$A \mapsto \overline{A}^T$ defines a semilin. transf. $Mat_{n\times n}(\mathbb{C}) \to Mat_{n\times n}(\mathbb{C})$.
Here σ = complex conjugation.

- Linear $\implies$ semilinear ($\sigma = 1$)
- $\ker S$ and $\operatorname{im} S$ are subspaces, $\dim V = \dim(\ker S) + \dim(\operatorname{im} S)$.
- σ is uniquely determined by S if $S \neq 0$, and $S \mapsto \sigma$ defines a surjecti

$$GL(V) \triangleleft \Gamma L(V) \longrightarrow Aut(k)$$

- $\Gamma L(V)$ takes subspaces to subspaces, so acts on $\mathcal{P}(V)$, and $P\Gamma L(V) := \Gamma L(V)/Z(V)$ acts faithfully on $\mathcal{P}(V)$

5 The Linear Groups

$GL(V) \cong GL_n(k)$ and $\Gamma L(V) \cong \Gamma L_n(k)$ act on $V \cong k^n$

$SL(V) = \ker(\det) : GL(V) \to k^\times$, $\qquad SL(V) \trianglelefteq GL(V)$,

$GL(V)/SL(V) \cong k^\times \cong GL_1(k)$ and $\Gamma L(V)/SL(V) \cong \Gamma L_1(k)$

$GL(V) \cong GL_n(k)$ (even $\Gamma L(V)$) acts on $\mathcal{P}(V) \cong \mathcal{P}(k^n)$, and $Z(V) \cong Z(k^n)$ is the subgroup acting trivially.

$PGL(V) \cong PGL_n(k)$ acts faithfully on $\mathcal{P}(V) \cong \mathcal{P}(k^n)$.

$PSL(V) \triangleleft PGL(V)$ and $PGL(V)/PSL(V) \cong k^\times/(k^\times)^n$.

6 Fundamental Theorem of Projective Geometry

A type-preserving projective transformation of $\mathcal{P}(V)$ is a bijection $\mathcal{P}(V) \to \mathcal{P}(V)$ carrying lines to lines and, more generally, m-subspaces to m-subspaces for each m.

Theorem

- *(F.T. of Proj. Geom.) If $n \geq 3$, then $P\Gamma L(V)$, considered as a group of permutations of $\mathcal{P}(V)$, is the group of all type-preserving projective transformations of $\mathcal{P}(V)$.*

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- *If $n = 2$, then $PGL(V) \cong PGL_2(k)$ is the group of all bijections of $\mathcal{P}(V) \cong \mathcal{P}^1(k) = k \cup \{\infty\}$ that preserve the cross-ratio*

$$(a; b; c; d) := \left(\frac{a-b}{a-c}\right) \Big/ \left(\frac{d-b}{d-c}\right).$$

$PGL(2, \mathbb{C})$ is the group of all continuous bijections of $\mathcal{P}^1(\mathbb{C})$ preserving (signed) angles. It also preserves the set of all circles and lines. The slightly larger group $PGL(2, \mathbb{C})\langle\sigma\rangle$, where $\sigma =$ complex conjugation, is the group of all continuous bijections preserving the set of all circles and lines. It preserves unsigned angles.

7 Simplicity Theorem for $PSL(V)$

Theorem

For $n > 1$, $PSL(V) \cong PSL_n(k)$ is simple unless $n = 2$ with $|k| \leq 3$.

Example

$PSL_2(2) = PGL_2(2) \cong Sym(3), \quad PSL_2(3) \cong Alt(4).$

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- $G_R := \begin{bmatrix} GL_{n-1}(k) & 0 \\ *\,*\,* & * \end{bmatrix}$, $Q := \begin{bmatrix} \mathrm{I} & 0 \\ *\,*\,* & 1 \end{bmatrix} \triangleleft G_R$. $\quad Q$ is abelian

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- If $N \leq G_R$, then $N = gNg^{-1}$ fixes $gR\ \forall\, g \in G \implies N = 1$
- Assume $N \not\leq G_R$. Doubl. transitiv. $\Longrightarrow G = NG_R \implies NQ \trianglelefteq G$.
- ♢ If $n \geq 3$, all "shearing" elem. matrices $\in NQ$; $\implies NQ = G$.
- $NQ = G \implies G/N \cong Q/N \cap Q$ abelian $\implies N$ contains all "commutators" $[g, h] := ghg^{-1}h^{-1}$, $\quad g, h \in G$

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- $(*)$ All $x \in Q$ are expressible as products of commutators

$$\implies Q \leq N \implies N = G.$$

8 Bigger group: Affine general linear group

Definition

A bijection $A : V \to V$ *is* *affine linear* $\iff \exists\, T \in GL(V),\ v_0 \in V$:

$$A(v) = T(v) + v_0 \quad \forall v \in V.$$

Equivalently, $A = \tau_{v_0} \circ T$, *where* $\tau_{v_0} : v \mapsto v + v_0 \quad \forall v \in V$

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- T and $v_0 = A(0)$ are uniquely determined by A.
- Define $Trans(V) = \{\tau_v \mid v \in V\} \cong V^+$. $\quad (\tau_v \tau_w = \tau_{v+w})$
  The homomorphism $A \mapsto T$ has kernel $Trans(V)$:
  $$Trans(V) \lhd AGL(V) \to GL(V).$$
  ♢
- $GL(V)$ is the subgroup of $AGL(V)$ fixing 0.
  $AGL(V) = Trans(V)GL(V)$, $\quad Trans(V) \cap GL(V) = 1$,
- Every subgroup X of $GL(V)$ has
  an affinized version $AX := Trans(V)X \leq AGL(V)$
  and a projectivized version $PX := XZ(V)/Z(V) \leq PGL(V)$.

9 The classical groups arise from the symmetry groups of certain geometries on V and $\mathcal{P}(V)$ – These geometries arise from BILINEAR, SESQUILINEAR, or QUADRATIC FORMS

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- A bilinear form on V is a function $f : V \times V \to k$ that is linear in each argument. Writing (v, w) in place of $f(v, w)$, this means that

$$\forall\ v, w, x \in V, \alpha, \beta \in k, \begin{cases} (1) & (\alpha v + \beta w, x) = \alpha(v, x) + \beta(w, x) \\ (2) & (x, \alpha v + \beta w) = \alpha(x, v) + \beta(x, w) \end{cases}$$

- When $k = \mathbb{C}$, f is a sesquilinear form iff it satisfies (1) and $(\bar{2})$ $(x, \alpha v + \beta w) = \overline{\alpha}(x, v) + \overline{\beta}(x, w)$, $\qquad \overline{a + bi} = a - bi$
- Sesquilinear forms over an arbitrary field k can be defined with respect to an automorphism $\alpha \mapsto \overline{\alpha}$ of k of order 2 that has been fixed in advance ($\overline{\overline{\alpha}} = \alpha$).

◊◊
- A quadratic form on V is a function $q : V \to k$ such that $q(\alpha v) =$

10 Examples of forms

- Dot product on $\mathbb{R}^n$: $([\alpha_1 \ \dots \ \alpha_n], [\beta_1 \ \dots \ \beta_n]) = \sum_{i=1}^n \alpha_i \beta_i$. This form is bilinear. It is also positive definite: $(v, v) > 0$ for all $v \neq 0$, and symmetric: $(v, w) = (w, v)$.
- Dot product on $\mathbb{C}^n$: $([\alpha_1 \ \dots \ \alpha_n], [\beta_1 \ \dots \ \beta_n]) = \sum_{i=1}^n \alpha_i \overline{\beta_i}$. This form is sesquilinear. It is also positive definite, and Hermitian symmetric: $(v, w) = \overline{(w, v)}$.
- Trace form on matrices is symmetric bilinear: $V = Mat_{n \times n}(k)$, $(A, B) = Tr(AB)$.
- Let $n > 2$ and consider the dot product above but use $F_2 \cong \mathbb{Z}/2\mathbb{Z}$ instead of $\mathbb{R}$. $W := \{[\alpha_1, \dots, \alpha_n] \mid \sum_i \alpha_i = 0\}$ is a subspace of V. As a form on W, the dot product is alternating: $(v, v) = 0$.
- Any alternating form is antisymmetric: $(v, w) = -(w, v)$. $(v, w) + (w, v) = (v + w, v + w) - (v, v) - (w, w) = 0$. Converse holds if $2 \neq 0$: $(v, v) = -(v, v) \implies 2(v, v) = 0$.
- V =space-time, coordinates x, y, z, t. The Lorentzian form is defined by the matrix $\text{diag}(1, 1, 1, -c^2)$.

11 Coordinatizing Forms

- Given a bilinear form f and any basis $B = (v_1, \ldots, v_n)$ of V,

$$\Big(\sum_i \alpha_i v_i, \sum_j \beta_j v_j\Big) = \sum_{i,j} \alpha_i \beta_j (v_i, v_j) = [\alpha_i] A [\beta_j]^T$$

  where $A = [(v_i, v_j)]_{i,j=1}^n =: (f)_B$.
- Conversely, given $n \times n$ A, $\big(\sum \alpha_i v_i\,, \sum_j \beta_j v_j\big) := [\alpha_i]_i A [\beta_j]_j^T$ defines a bilinear form on V.

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- Conversely, given $n \times n$ A, $\left(\sum \alpha_i v_i \, , \, \sum_j \beta_j v_j\right) := [\alpha_i]_i A [\beta_j]_j^T$ defines a bilinear form on V.
- Given V and B, this sets up a bijection

$$Mat_{n\times n}(k) \longleftrightarrow \text{the set of all bilinear forms on } V.$$

- For sesquilinear forms, replace $[\beta_j]^T$ by $\overline{[\beta_j]}^T$.
- f is $\begin{cases} \text{symmetric} & \iff A = A^T \\ \text{alternating} & \iff A = -A^T \text{ and } a_{ii} = 0 \ \forall i \\ \text{Hermitian-symmetric} & \iff A = \overline{A}^T \end{cases}$

With respect to another basis C, get $(f)_C = M_{B,C}(f)_B M_{B,C}^T$.

12 Isometries

Definition

Let V and W be vector spaces over k, with bilinear forms f and g respectively. An ***isometry*** *from V to W is an invertible linear transformation $T : V \to W$ such that*

$$g(Tv, Tw) = (v, w), \quad \forall v, w \in V.$$

♢

Write $f \sim g$ if such an isometry exists ($V \sim W$, if f, g understood).

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- Composites and inverses of isometries are isometries.
- The relation of being isometric is an equivalence relation on vector spaces with forms.
- If $f \sim g$ and f is symmetric (or alternating) then so is g.
- With respect to fixed bases of V and W, suppose that f has matrix A, g has matrix B, and T has matrix X. Then
  $$g(Tv, Tw) = (v, w)\ \forall\, v, w \in V \iff B = XAX^T.$$
  Write $A \approx B \iff \exists\,\text{invertible}\, X : \quad B = XAX^T$.
- Similar definition can be made for sesquilinear forms; but $B = XA\overline{X}^T$, above.
- $\sim$ classes of forms on $V \leftrightarrow \approx$ classes on $Mat_{n\times n}(k)$.

13 Orthogonality and nondegeneracy

Assume we are given V, a basis B, and a bilinear or sesquilinear form such that $(v, w) = 0 \iff (w, v) = 0$.
For any $v, w \in V$, write $v \perp w \iff (v, w) = 0$, a symmetric relationn.

◇ For any $X \subseteq V$, set $X^{\perp} = \{w \in V \mid v \perp w \text{ for all } v \in X\}$.

Orthogonality and nondegeneracy

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Lemma-Definition (Nondegeneracy of a form)

♢

Given a bilinear or sesquilinear form f on V, having matrix A with respe

- *f is nondegenerate.*
- $V^{\perp} = 0$.
- $\det A \neq 0$.
- $\dim X + \dim X^{\perp} = n$ *for all subspaces* $X \subseteq V$. *(always $\geq n$)*
- $X = X^{\perp\perp}$ *for all subspaces* $X \subseteq V$. *(always $\subseteq$)*

Remarks

- A degenerate example: Let X be the vector space of 0-1 words of length m, with an even number of 1's. Assume that m is even. Then $V^{\perp} = \{0, \langle 1\ \ldots\ 1\rangle\}$.
- By abuse of language, if f is nondegenerate, then we also say that V is nondegenerate. If X is a subspace of V, we say that X is nondegenerate if and only if the restriction of f to X is nondegenerate. If form is pos. def., then all subspaces are nondegenerate.
- If X is a nondegenerate subspace of V, then $X \cap X^{\perp} = 0$. If V is also nondegenerate, then $\dim X + \dim X^{\perp} = \dim V$, so $V = X \oplus X^{\perp} =: X \perp X^{\perp}$. And then $X^{\perp}$ is nondegenerate since $X^{\perp\perp} = X$!
  Such "nondegenerate" or "orthogonal" decompositions of V enable inductive proofs of some geometric properties of V.
- A key to the equivalences in the lemma-definition is the (semi)linear mapping

$$\phi : V \to V^* := Hom(V, k) \quad \text{defined by } (\phi(v))(w) := (w, v).$$

15 AND NOW THE STARS OF OUR SHOW . . .

♢♢

- If f is nondegenerate bilinear symmetric, $O(V,f) := \{T \in GL(V) \mid$
- If f is nondegenerate bilinear alternating, $Sp(V,f) := \{T \in GL(V) \mid$
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Let B be a basis, A the matrix of f with respect to B. Assume that A is invertible (f is nondegenerate).

- For $A^T = A$,
  $O_n(k) = O_n(k, A) := \{X \in GL_n(k) \mid XAX^T = A\}$
- For $A^T = -A$, all $a_{ii} = 0$ all i,
  $Sp_n(k) = Sp_n(k, A) := \{X \in GL_n(k) \mid XAX^T = A\}$
- For $\overline{A}^T = A$,
  $U_n(k) = U_n(k, A) := \{X \in GL_n(k) \mid XA\overline{X}^T = A\}$

16 $O(n) := O(\mathbb{R}^n, f)$, $f =$ dot product

- There exists an orthonormal basis (o.b.) of $\mathbb{R}^n$; the corresponding matrix A of f is $A = I$.
- Then the columns of any $X \in O(n)$ form an o.b. of $\mathbb{R}^n$.
- $O(n) \leftrightarrow$ set of all o.b.'s of V; $O(n)$ is **compact**.
- $XAX^T = A \implies \det X^2 = 1 \implies \det X = \pm 1$.

◊
- diag$([\pm 1, \pm 1, \ldots, \pm 1]) \in O(n)$, and the subgroup $SO(n) = O(n) \cap S$

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- $O(1) = \langle \pm 1 \rangle$, $SO(1) = 1$.

♢ ▸

$O(2) = S^1 \langle \iota \rangle$, $SO(2) = S^1$ =group of rotations $\cong \mathbb{R}^+/2\pi Z$, ι =refl

- $O(3) = SO(3) \times \langle -1 \rangle$.
  Principal axis theorem: Every element of $SO(3)$ is a rotation.
- $SO(3) - \{1\} \longrightarrow \mathcal{P}(\mathbb{R}^3) =: \mathcal{P}^2(\mathbb{R}), X \mapsto$ axis of X,
  fibers are copies of $S^1 - \{1\}$.

Orthogonal groups, ctd

Theorem (Symmetric forms are diagonalizable except in char. 2)

Assume that $2 \neq 0$ in k. Then for any symmetric bilinear form on V, there exists an orthogonal basis B, i.e., $(v_i, v_j) = 0$ for all $i \neq j$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Namely, if $(v_1, v_1) = 0 = (v_2, v_2)$, $(v_1, v_2) = 1$, then put $w_\pm := v_1 \pm v_2$ to get $(w_+, w_-) = 0$, $(w_+, w_+) = 2$, $(w_-, w_-) = -2$.

Theorem (Real case – Sylvester's Law/Theorem)

Let $V = \mathbb{R}^n$ be equipped with a nondegenerate symmetric bilinear form. Then there exists an orthogonal basis $B = \{v_1, \ldots, v_n\}$ of V and an integer $0 \leq a \leq n$ independent of the basis such that $(v_1, v_1) = \cdots = (v_a, v_a) = 1$ and $(v_{a+1}, v_{a+1}) = \cdots = (v_n, v_n) = -1$.

Equivalently, for any real invertible symmetric A there is a real X such that $XAX^T = diag[1, \ldots, 1, -1, \ldots, -1]$, and the number of 1's is independent of X.

Symplectic groups

If f is alternating and nondegenerate, pick any $0 \neq v \in V$. Then $v \notin V^{\perp}$ so $\exists w \in V: \ (v,w) = 1, \ (w,v) = -1$.
And $W = kv + kw$ is nondegenerate, so $V = W \oplus W^{\perp}$. On W, the form has matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Such a 2-dimensional subspace is called a **hyperbolic plane**.

Theorem (One alternating form only)

If f is nondegenerate and alternating, then V is the orthogonal sum of hyperbolic planes. In particular $\dim V$ *is even.*

The notations $Sp(V)$ and $Sp(2n,k)$ are common. When $k = \mathbb{R}$, $Sp(2n)$ and $S(2n)$ are used. $Sp(2n)$ is **not compact**.

Theorem

$Sp(V) \leq SL(V)$, *i.e.,* $\det X = 1$ *for all* $X \in Sp(V)$. *Moreover* $Sp(V) \cap Z(V) = \langle \pm 1 \rangle$. *If* $\dim V = 2$, *then* $Sp(V) = SL(V)$.

Unitary groups

With respect to complex conjugation, for any positive definite hermitian symmetric form on $V = \mathbb{C}^n$, there is an orthonormal basis.

If $X \in U(n)$, then $X\overline{X}^T = I$, and so $\delta := \det X$ satisfies $\delta\overline{\delta} = 1$, so $\delta \in S^1$.

Like $O(n)$, $U(n)$ is compact.

A Spin Covering Example: $SU(2) \longrightarrow SO(3)$

Start with the real vector space

$$\mathbf{u} := \left\{ \begin{bmatrix} a & b+ci \\ b-ci & -a \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\} = \{X \mid \overline{X}^T = X \text{ and } Tr(X) = 0\}.$$

Then $\dim_{\mathbb{R}}(\mathbf{u}) = 3$, and the form $(X, Y) = Tr(XY)/2$ is symmetric bilinear, with $(X, X) = a^2 + b^2 + c^2$.
Thus, $\mathbf{u} \sim \mathbb{R}^{\mathbf{3}}$ via $X \mapsto [a\, b\, c]$.

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Now bring in $SU(2)$. If $X \in \mathbf{u}$ and $U \in SU(2)$, write

$$c_U(X) = UXU^{-1} = UX\overline{U}^T$$

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$$Tr((c_U X)^2) = Tr(c_U(X^2)) = Tr(X^2) \implies c_U \in O(\mathbf{u}) \cong O(3)$$

$c_V(c_U(X)) = VUXU^{-1}V^{-1} = c_{VU}(X)$, so π is a homomorphism:

$$\pi : SU(2) \longrightarrow O(\mathbf{u}) \cong O(3), \qquad U \mapsto c_U$$

One computes $\ker \pi = \langle \pm 1 \rangle$, and $\operatorname{im} \pi = SO(\mathbf{u})$.
Every orthogonal group has a “spin covering” $Spin(V) \to SO(V)$.
Here, $Spin(3) \cong SU(2)$.

21 More on simplicity

◇

Most classical groups have a simple "core" like $PSL(V)$, in between an al

One key to the core's simplicity is the geometric homogeneity.

Theorem (Witt's Lemma)

Let V have a nondegenerate symmetric, alternating, or Hermitian-symmetric form. Let W and W' be isometric subspaces of V. Then any isometry $T : W \to W'$ extends to an isometry T^* of V.

$$\begin{array}{ccc} W & \xrightarrow{T} & W' \\ \downarrow i & & \downarrow i \\ V & \xrightarrow{T^*} & V \end{array}$$

◇ Over an algebraically closed field, in addition to the classical groups – L, S, O – there are exactly 5 simple algebraic groups. Over $\mathbb{C}$ the same is true for simple Lie groups.

Over a finite field F or $F = \mathbb{R}$, each of the classical groups over the algebraic closure $\overline{F}$ or $\mathbb{C}$ gives rise to one or more "F-forms," which are finite groups or real Lie groups (resp.) with a simple core as above, e.g. $SU(n)$ and $SL_n(\mathbb{R})$ are two forms of $SL_n(\mathbb{C})$.

THE END

22 Complements

- p.2 $\mathbf{P}^2(\mathbb{R})$ can be obtained from the sphere S^2 (i.e., the surface of the Earth) by identifying each point with its antipode (opposite).
- p.3 Each column of $M_{C,B}$ consists of the coordinates of one vector in C with respect to the basis B.
  $PGL(V)/PSL(V) \cong k^\times/(k^\times)^n$, where $(k^\times)^n$ is the multiplicative group of n^{th} powers of elements of $k^\times$.
  $Z(V) \cap SL(V) = Z(SL(V))$ is isomorphic to the group of n^{th} roots of unity in k.
- p.3 One can define the groups GL, SL, PGL, etc., using a ring Y instead of a field k. The group structure is much more complicated in this case. Such groups (particularly $PSL_2(\mathbb{Z})$) are fundamental in number theory.
- p.4 There are complements to $GL(V)$ in $\Gamma L(V)$. Fix a basis B of V. For any $\sigma \in Aut(k)$, $\sigma_B(\sum \alpha_i v_i) = \sum \sigma(\alpha_i) v_i$ defines $\sigma_B \in \Gamma L(V)$. Then $H_B := \{\sigma_B \,|\, \sigma \in Aut(k)\}$ is a complement to $GL(V)$.

23 Complements, c't'd

- p.3 The **fractional linear transformation**
$$z \mapsto \frac{az+b}{cz+d}$$
is induced on $\mathcal{P}^1(k) = k \cup \{\infty\}$ by applying the element of $PGL_2(k)$ corresponding to $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(k)$.
- p.3 In the case $n = 2$, the group $PGL(V)$ is **sharply triply transitive** on $\mathcal{P}(V) = \mathcal{P}^1(k)$: for any six points $P, Q, R, P', Q', R' \in \mathcal{P}(V)$ such that P, Q, R are distinct and P', Q', R' are distinct, there is a unique $g \in PGL(V)$ such that $gP = P'$, $gQ = Q'$, and $gR = R'$.
- p.7 Simplicity when $n = 2$. The proof doesn't cover the case $n = 2$, $G \neq NQ$. But Since $GL_1(k) \cong k^\times$ is abelian, G/NQ is abelian. Repeat the final two steps with NQ in place of N to reach the contradiction $NQ = G$. The condition $|k| > 3$ is required for Q to be generated by commutators. Commutators of the form $[g, h]$, g diagonal, $h \in Q$, do the trick.

Complements, c't'd

- p.8, 16 The group of all rigid (i.e., distance-preserving) motions of $\mathbb{R}^n$ is the affine orthogonal group $AO(n)$.
- p.9 In the study of quadratic forms over fields of characteristic 2 one often assumes that every element of the field is a square.
- p.9 If $2 \neq 0$, then a symmetric form $(\ ,\ )$ gives rise to a quadratic form q by $q(v) = (v, v)$. If $2 \neq 0$, the quadratic form gives back the symmetric form as $(v, w) = \frac{1}{2}\left[q(v+w) - q(v) - q(w)\right]$, so q and $(\ ,\ )$ determine each other. If $2 = 0$, then $(\ ,\ )$ gives rise to q as before, but q is degenerate: $q(v+w) - q(v) - q(w) = 0$.
- p.12 If $2 \neq 0$, and f and g are symmetric, then the isometry condition is equivalent to
$$g(Tv, Tv) = f(v, v) \text{ for all } v \in V.$$
- p.13 A quadratic form q is called nondegenerate $\iff$ its corresponding symmetric bilinear form is nondegenerate.

25 Complements, c't'd

- p.13 $X^\perp$ is always a subspace of V.
- p.13 (2)–(5) are equivalent: $\dim V = \dim V^*$ (V is fin-dim), and $V^\perp = \ker \phi$. So (2) $\iff$ (3) $\iff$ ϕ is an isomorphism. Define $\rho_X : V^* \to X^*$ by $\rho_X(f) = f|_X$ (restriction to X); ρ_X is surjective. If ϕ is an isomorphism, then $\psi := \rho_X \circ \phi$ is surjective and $n = \dim \ker \psi + \dim X^* = \dim X^\perp + \dim X$. Therefore (2) $\implies$ (4). By dimension count, (4) $\implies$ (5). Finally take $X = 0$ in (5) to get (1).
- p.15 The following theorem can be found in Artin's *Geometric Algebra*: Any nondegenerate bilinear form on V with the property that $(v, w) = 0 \implies (w, v) = 0$ must be either symmetric or alternating.
- p.15 Only when there is no danger of confusion should f or A be omitted from the group notation.

Complements, c't'd

- p.16 $O(2)$ is a continuous analogue of the finite dihedral groups, all of which lie in $O(2)$ and whose union is a dense subgroup. To get the finite dihedral group of order $2n$, replace S^1 by the cyclic group of rotations through multiples of $2\pi/n$.
- p.16 By contrast with the case of $PSL(V)$, which has no nontrivial abelian quotients (with 2 exceptions), $SO(V)$ has a homomorphism onto an abelian group of exponent 2, often surjective: "spinorial norm" $SO(V) \to k^\times/(k^\times)^2$ (char $k \neq 2$), and "Dickson invariant" $SO(V) \to Z_2$ (char $k = 2$)
- p.6, 21 Just as $PGL(V)$ has a natural faithful action on $\mathcal{P}(V)$, every simple classical group and indeed every "simple group of Lie type" G has a natural faithful action on a geometry $B(G)$ called a (Tits) building. Exception must be made for "rank 1" groups e.g. $PSL_2(k)$. The grps $Aut(G)$ and $Aut(B(G))$ coincide, when $Aut(B(G))$ is understood to include all non-type-preserving automorphisms of $B(G)$ too. This generalizes FT Proj. Geom. Today Kac-Moody groups are studied, associated with twin buildings.