

The Classical Groups

IMR

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1 References

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Fixed notation:

V is an n -dimensional vector space over a field k and $\mathcal{P}(V) = \mathcal{P}^{n-1}(k) =$ the corresponding projective space

The **points** of $\mathcal{P}(V)$ are the 1-dimensional subspaces of V .

$$\begin{aligned} \Phi : V - \{0\} &\longrightarrow \mathcal{P}(V) \\ v &\longmapsto kv \end{aligned}$$

A **line** (resp. **m -subspace** of $\mathcal{P}(V)$) is the image under Φ of a 2-dimensional (resp. $m + 1$ -dimensional) subspace of V .

- ▶ The projective line: $\mathcal{P}^1(k) = k \cup \{\infty\}$ (think slopes)
- ◊ ▶ The projective plane: $\mathcal{P}^2(k) = k^2 \cup \ell_\infty$,
 $\ell_\infty \cong \mathcal{P}^1(k) =$ "line at infinity"
- ▶ $\mathcal{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} =$ "Riemann sphere"

3 Groups of symmetries of V and $\mathcal{P}(V)$

- ◇ ▶ $GL(V) := \{T : V \rightarrow V \mid T \text{ is an invertible lin. transf.}\}$
... $GL_n(k) = GL(n, k) := \{X \in Mat_{n \times n}(k) \mid X \text{ is invertible}\}$
 \forall ordered basis $B = \{v_1, \dots, v_n\}$ there is a group isomorphism $GL(V) \cong GL_n(k)$, $T \leftrightarrow [T]_B = \text{matrix of } T \text{ with resp. to } B$
 $\det T := \det[T]_B$ is a good definition; for any other basis C ,
- ◇ $[T]_C = M_{C,B}[T]_B M_{C,B}^{-1}$, $M_{C,B} = \text{"change of basis" matrix}$
 - ▶ $SL(V) := \{T \in GL(V) \mid \det T = 1\}$
 $\cong SL_n(k) := \{X \in GL_n(k) \mid \det X = 1\}$
 - ▶ $Z(V) := \{\gamma 1_V \mid \gamma \in k^\times\} \cong Z_n(k) := \{\gamma I \mid \gamma \in k^\times\} \cong k^\times$

Lemma

Let $T \in GL(V)$. Then $T \in Z(V) \iff T$ fixes $\mathcal{P}(V)$ pointwise.

- ◇◇ ▶ $PGL(V) := GL(V)/Z(V)$ acts **faithfully** on $\mathcal{P}(V)$
 $PGL_n(k) := GL_n(k)/Z_n(k)$, etc.
- ▶ $PSL(V) = SL(V)Z(V)/Z(V) \cong SL(V)/Z(V) \cap SL(V)$,
 $PSL(V) \trianglelefteq PGL(V)$.

4 Bigger groups: $\Gamma L(V)$ and $P\Gamma L(V)$

Definition

$S : V \rightarrow V$ is a **semilinear transformation** $\iff \exists \sigma \in \text{Aut}(k)$
 $S(\alpha v + \beta w) = \sigma(\alpha)S(v) + \sigma(\beta)S(w), \quad v, w \in V, \quad \alpha, \beta \in k.$

$$\Gamma L(V) = \{S : V \rightarrow V \mid S \text{ is invertible semilin. transf.}\}$$

Example (“Hermitian transpose”)

$A \mapsto \overline{A}^T$ defines a semilin. transf. $\text{Mat}_{n \times n}(\mathbb{C}) \rightarrow \text{Mat}_{n \times n}(\mathbb{C})$.
Here $\sigma =$ complex conjugation.

- ▶ Linear \implies semilinear ($\sigma = 1$)
- ▶ $\ker S$ and $\text{im } S$ are subspaces, $\dim V = \dim(\ker S) + \dim(\text{im } S)$.
- ◇ ▶ σ is uniquely determined by S if $S \neq 0$, and $S \mapsto \sigma$ defines a surjective homomorphism with kernel $GL(V)$:

$$GL(V) \triangleleft \Gamma L(V) \longrightarrow \text{Aut}(k)$$

- ▶ $\Gamma L(V)$ takes subspaces to subspaces, so acts on $\mathcal{P}(V)$, and $P\Gamma L(V) := \Gamma L(V)/Z(V)$ acts faithfully on $\mathcal{P}(V)$.

5 The Linear Groups

$GL(V) \cong GL_n(k)$ and $\Gamma L(V) \cong \Gamma L_n(k)$ act on $V \cong k^n$

$SL(V) = \ker(\det) : GL(V) \rightarrow k^\times$, $SL(V) \trianglelefteq GL(V)$,

$GL(V)/SL(V) \cong k^\times \cong GL_1(k)$ and $\Gamma L(V)/SL(V) \cong \Gamma L_1(k)$

$GL(V) \cong GL_n(k)$ (even $\Gamma L(V)$) acts on $\mathcal{P}(V) \cong \mathcal{P}(k^n)$, and $Z(V) \cong Z(k^n)$ is the subgroup acting trivially.

$PGL(V) \cong PGL_n(k)$ acts faithfully on $\mathcal{P}(V) \cong \mathcal{P}(k^n)$.

$PSL(V) \triangleleft PGL(V)$ and $PGL(V)/PSL(V) \cong k^\times / (k^\times)^n$.

6 Fundamental Theorem of Projective Geometry

A **type-preserving projective transformation** of $\mathcal{P}(V)$ is a bijection $\mathcal{P}(V) \rightarrow \mathcal{P}(V)$ carrying lines to lines and, more generally, m -subspaces to m -subspaces for each m .

Theorem

- ◇ ▶ (F.T. of Proj. Geom.) If $n \geq 3$, then $P\Gamma L(V)$, considered as a group of permutations of $\mathcal{P}(V)$, is the group of **all** type-preserving projective transformations of $\mathcal{P}(V)$.
- ▶ If $n = 2$, then $PGL(V) \cong PGL_2(k)$ is the group of all bijections of $\mathcal{P}(V) \cong \mathcal{P}^1(k) = k \cup \{\infty\}$ that preserve the **cross-ratio**

$$(a; b; c; d) := \left(\frac{a-b}{a-c} \right) / \left(\frac{d-b}{d-c} \right).$$

$PGL(2, \mathbb{C})$ is the group of all continuous bijections of $\mathcal{P}^1(\mathbb{C})$ preserving (signed) angles. It also preserves the set of all circles and lines. The slightly larger group $PGL(2, \mathbb{C})\langle\sigma\rangle$, where $\sigma =$ complex conjugation, is the group of **all** continuous bijections preserving the set of all circles and lines. It preserves unsigned angles.

7 Simplicity Theorem for $PSL(V)$

Theorem

For $n > 1$, $PSL(V) \cong PSL_n(k)$ is simple unless $n = 2$ with $|k| \leq 3$.

Example

$PSL_2(2) = PGL_2(2) \cong Sym(3)$, $PSL_2(3) \cong Alt(4)$.

- ◇ Sketch (for $n \geq 3$): let $G = PSL_n(k)$ and $N \trianglelefteq G$.
- ▶ G is doubly transitive on $\mathcal{P}(V)$. Fix $R = k[0 \cdots 0 1]^T \in \mathcal{P}(V)$.
 - ▶ $G_R := \begin{bmatrix} GL_{n-1}(k) & 0 \\ *** & * \end{bmatrix}$, $Q := \begin{bmatrix} I & 0 \\ *** & 1 \end{bmatrix} \triangleleft G_R$. Q is abelian
 - ▶ If $N \leq G_R$, then $N = gNg^{-1}$ fixes $gR \forall g \in G \implies N = 1$
 - ▶ Assume $N \not\leq G_R$. Doubl. transitiv. $\implies G = NG_R \implies NQ \trianglelefteq G$.
 - ▶ If $n \geq 3$, all "shearing" elem. matrices $\in NQ$; $\implies NQ = G$.
 - ▶ $NQ = G \implies G/N \cong Q/N \cap Q$ abelian $\implies N$ contains all "commutators" $[g, h] := ghg^{-1}h^{-1}$, $g, h \in G$
 - ▶ (*) All $x \in Q$ are expressible as products of commutators
 $\implies Q \leq N \implies N = G$.

8 Bigger group: Affine general linear group

Definition

A bijection $A : V \rightarrow V$ is **affine linear** $\iff \exists T \in GL(V), v_0 \in V :$

$$A(v) = T(v) + v_0 \quad \forall v \in V.$$

Equivalently, $A = \tau_{v_0} \circ T$, where $\tau_{v_0} : v \mapsto v + v_0 \quad \forall v \in V$

$$AGL(V) = \{A : V \rightarrow V \mid A \text{ is an affine linear bijection}\}$$

- ▶ T and $v_0 = A(0)$ are uniquely determined by A .
- ▶ Define $Trans(V) = \{\tau_v \mid v \in V\} \cong V^+$. ($\tau_v \tau_w = \tau_{v+w}$)
The homomorphism $A \mapsto T$ has kernel $Trans(V)$:
$$Trans(V) \triangleleft AGL(V) \rightarrow GL(V).$$
- ▶ $GL(V)$ is the subgroup of $AGL(V)$ fixing 0.
 $AGL(V) = Trans(V)GL(V), \quad Trans(V) \cap GL(V) = 1,$
- ▶ Every subgroup X of $GL(V)$ has
an affinized version $AX := Trans(V)X \leq AGL(V)$
and a projectivized version $PX := XZ(V)/Z(V) \leq PGL(V)$.

9 The classical groups arise from the symmetry groups of certain geometries on V and $\mathcal{P}(V)$ – These geometries arise from BILINEAR, SESQUILINEAR, or QUADRATIC FORMS

- ▶ A **bilinear form on V** is a function $f : V \times V \rightarrow k$ that is linear in each argument. Writing (v, w) in place of $f(v, w)$, this means that

$$\forall v, w, x \in V, \alpha, \beta \in k, \begin{cases} (1) & (\alpha v + \beta w, x) = \alpha(v, x) + \beta(w, x) \\ (2) & (x, \alpha v + \beta w) = \alpha(x, v) + \beta(x, w) \end{cases}$$

- ▶ When $k = \mathbb{C}$, f is a **sesquilinear form** iff it satisfies (1) and $(\bar{2})$ $(x, \alpha v + \beta w) = \bar{\alpha}(x, v) + \bar{\beta}(x, w)$, $\overline{a + bi} = a - bi$
- ▶ Sesquilinear forms over an arbitrary field k can be defined with respect to an automorphism $\alpha \mapsto \bar{\alpha}$ of k of order 2 that has been fixed in advance ($\bar{\bar{\alpha}} = \alpha$).

- ◆◆ ▶ A **quadratic form** on V is a function $q : V \rightarrow k$ such that $q(\alpha v) = \alpha^2 q(v)$, and $f(v, w) := q(v + w) - q(v) - q(w)$ is bilinear. These are needed only for fields k in which $2 = 0$.

10 Examples of forms

- ▶ Dot product on \mathbb{R}^n : $([\alpha_1 \dots \alpha_n], [\beta_1 \dots \beta_n]) = \sum_{i=1}^n \alpha_i \beta_i$.
This form is bilinear. It is also positive definite: $(v, v) > 0$ for all $v \neq 0$, and **symmetric**: $(v, w) = (w, v)$.
- ▶ Dot product on \mathbb{C}^n : $([\alpha_1 \dots \alpha_n], [\beta_1 \dots \beta_n]) = \sum_{i=1}^n \alpha_i \overline{\beta_i}$.
This form is sesquilinear. It is also positive definite, and **Hermitian symmetric**: $(v, w) = \overline{(w, v)}$.
- ▶ Trace form on matrices is symmetric bilinear:
 $V = \text{Mat}_{n \times n}(k)$, $(A, B) = \text{Tr}(AB)$.
- ▶ Let $n > 2$ and consider the dot product above but use $F_2 \cong \mathbb{Z}/2\mathbb{Z}$ instead of \mathbb{R} . $W := \{[\alpha_1, \dots, \alpha_n] \mid \sum_i \alpha_i = 0\}$ is a subspace of V . As a form on W , the dot product is **alternating**: $(v, v) = 0$.
- ▶ Any alternating form is antisymmetric: $(v, w) = -(w, v)$.
 $(v, w) + (w, v) = (v + w, v + w) - (v, v) - (w, w) = 0$.
Converse holds if $2 \neq 0$: $(v, v) = -(v, v) \implies 2(v, v) = 0$.
- ▶ $V = \text{space-time}$, coordinates x, y, z, t . The Lorentzian form is defined by the matrix $\text{diag}(1, 1, 1, -c^2)$.

11 Coordinatizing Forms

- ▶ Given a bilinear form f and any basis $B = (v_1, \dots, v_n)$ of V ,

$$\left(\sum_i \alpha_i v_i, \sum_j \beta_j v_j \right) = \sum_{i,j} \alpha_i \beta_j (v_i, v_j) = [\alpha_i] A [\beta_j]^T$$

where $A = [(v_i, v_j)]_{i,j=1}^n =: (f)_B$.

- ▶ Conversely, given $n \times n$ A , $(\sum_i \alpha_i v_i, \sum_j \beta_j v_j) := [\alpha_i]_i A [\beta_j]_j^T$ defines a bilinear form on V .
- ▶ Given V and B , this sets up a bijection

$$\text{Mat}_{n \times n}(k) \longleftrightarrow \text{the set of all bilinear forms on } V.$$

- ▶ For sesquilinear forms, replace $[\beta_j]^T$ by $[\overline{\beta_j}]^T$.

- ▶ f is $\begin{cases} \text{symmetric} & \iff A = A^T \\ \text{alternating} & \iff A = -A^T \text{ and } a_{ii} = 0 \forall i \\ \text{Hermitian-symmetric} & \iff A = \overline{A}^T \end{cases}$

With respect to another basis C , get $(f)_C = M_{B,C}(f)_B M_{B,C}^T$.

12 Isometries

Definition

Let V and W be vector spaces over k , with bilinear forms f and g respectively. An **isometry** from V to W is an invertible linear transformation $T : V \rightarrow W$ such that

$$g(Tv, Tw) = (v, w), \quad \forall v, w \in V.$$

Write $f \sim g$ if such an isometry exists ($V \sim W$, if f, g understood).

- ▶ Composites and inverses of isometries are isometries.
- ▶ The relation of being isometric is an equivalence relation on vector spaces with forms.
- ▶ If $f \sim g$ and f is symmetric (or alternating) then so is g .
- ▶ With respect to fixed bases of V and W , suppose that f has matrix A , g has matrix B , and T has matrix X . Then

$$g(Tv, Tw) = (v, w) \quad \forall v, w \in V \iff B = XAX^T.$$

$$\text{Write } A \approx B \iff \exists \text{ invertible } X : B = XAX^T.$$

- ▶ Similar definition can be made for sesquilinear forms; but $B = XAX^T$, above.

13 Orthogonality and nondegeneracy

Assume we are given V , a basis B , and a bilinear or sesquilinear form such that $(v, w) = 0 \iff (w, v) = 0$.

For any $v, w \in V$, write $v \perp w \iff (v, w) = 0$, a symmetric relation.

- ◇ For any $X \subseteq V$, set $X^\perp = \{w \in V \mid v \perp w \text{ for all } v \in X\}$.

Lemma-Definition (Nondegeneracy of a form)

- ◇ Given a bilinear or sesquilinear form f on V , having matrix A with respect to some basis B of V as above, the following conditions are equivalent:
 - ▶ f is *nondegenerate*.
 - ▶ $V^\perp = 0$.
 - ▶ $\det A \neq 0$.
 - ▶ $\dim X + \dim X^\perp = n$ for all subspaces $X \subseteq V$. (always $\geq n$)
 - ▶ $X = X^{\perp\perp}$ for all subspaces $X \subseteq V$. (always \subseteq)

14 Remarks

- ▶ A degenerate example: Let X be the vector space of 0-1 words of length m , with an even number of 1's. Assume that m is even. Then $V^\perp = \{0, \langle 1 \dots 1 \rangle\}$.
- ▶ By abuse of language, if f is nondegenerate, then we also say that V is nondegenerate. If X is a subspace of V , we say that X is nondegenerate if and only if the restriction of f to X is nondegenerate. If form is pos. def., then all subspaces are nondegenerate.
- ▶ If X is a nondegenerate subspace of V , then $X \cap X^\perp = 0$. If V is also nondegenerate, then $\dim X + \dim X^\perp = \dim V$, so $V = X \oplus X^\perp =: X \perp X^\perp$. And then X^\perp is nondegenerate since $X^{\perp\perp} = X$!

Such “nondegenerate” or “orthogonal” decompositions of V enable inductive proofs of some geometric properties of V .

- ▶ A key to the equivalences in the lemma-definition is the (semi)linear mapping

$$\phi : V \rightarrow V^* := \text{Hom}(V, k) \quad \text{defined by } (\phi(v))(w) := (w, v).$$

15 AND NOW THE STARS OF OUR SHOW ...

- ◆◆ ▶ If f is nondegenerate bilinear symmetric,
 $O(V, f) := \{T \in GL(V) \mid (Tv, Tw) = (v, w) \forall v, w \in V\}$
- ▶ compfourIf f is nondegenerate bilinear alternating,
 $Sp(V, f) := \{T \in GL(V) \mid (Tv, Tw) = (v, w) \forall v, w \in V\}$
- ▶ compfourIf f is nondegenerate sesquilinear hermitian-symmetric,
 $U(V, f) := \{T \in GL(V) \mid (Tv, Tw) = (v, w) \forall v, w \in V\}$

Let B be a basis, A the matrix of f with respect to B . Assume that A is invertible (f is nondegenerate).

- ▶ For $A^T = A$,
 $O_n(k) = O_n(k, A) := \{X \in GL_n(k) \mid XAX^T = A\}$
- ▶ For $A^T = -A$, all $a_{ii} = 0$ all i ,
 $Sp_n(k) = Sp_n(k, A) := \{X \in GL_n(k) \mid XAX^T = A\}$
- ▶ For $\bar{A}^T = A$,
 $U_n(k) = U_n(k, A) := \{X \in GL_n(k) \mid XAX^T = A\}$

16 $O(n) := O(\mathbb{R}^n, f)$, $f = \text{dot product}$

- ▶ There exists an orthonormal basis (o.b.) of \mathbb{R}^n ; the corresponding matrix A of f is $A = I$.
 - ▶ Then the columns of any $X \in O(n)$ form an o.b. of \mathbb{R}^n .
 - ▶ $O(n) \leftrightarrow$ set of all o.b.'s of V ; $O(n)$ is **compact**.
 - ▶ $XAX^T = A \implies \det X^2 = 1 \implies \det X = \pm 1$.
- ◇ ▶ $\text{diag}([\pm 1, \pm 1, \dots, \pm 1]) \in O(n)$, and the subgroup $SO(n) = O(n) \cap SL(\mathbb{R}^n)$ satisfies $[O(n) : SO(n)] = 2$.
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- ▶ $O(1) = \langle \pm 1 \rangle$, $SO(1) = 1$.
 - ◇ ▶ $O(2) = S^1 \langle \iota \rangle$, $SO(2) = S^1 = \text{group of rotations} \cong \mathbb{R}^+ / 2\pi\mathbb{Z}$,
 $\iota = \text{reflection in (say) x-axis}$, $\iota^2 = 1$, $\iota g \iota = g^{-1}$, $g \in S^1$.
 - ▶ $O(3) = SO(3) \times \langle -1 \rangle$.
Principal axis theorem: Every element of $SO(3)$ is a rotation.
 - ▶ $SO(3) - \{1\} \longrightarrow \mathcal{P}(\mathbb{R}^3) =: \mathcal{P}^2(\mathbb{R})$, $X \mapsto \text{axis of } X$,
fibers are copies of $S^1 - \{1\}$.

17 Orthogonal groups, ctd

Theorem (Symmetric forms are diagonalizable except in char. 2)

Assume that $2 \neq 0$ in k . Then for any symmetric bilinear form on V , there exists an orthogonal basis B , i.e., $(v_i, v_j) = 0$ for all $i \neq j$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \cdot \text{Namely, if } (v_1, v_1) = 0 = (v_2, v_2), (v_1, v_2) = 1, \text{ then put } w_{\pm} := v_1 \pm v_2 \text{ to get } (w_+, w_-) = 0, (w_+, w_+) = 2, (w_-, w_-) = -2.$$

Theorem (Real case – Sylvester's Law/Theorem)

Let $V = \mathbb{R}^n$ be equipped with a nondegenerate symmetric bilinear form. Then there exists an orthogonal basis $B = \{v_1, \dots, v_n\}$ of V and an integer $0 \leq a \leq n$ independent of the basis such that $(v_1, v_1) = \dots = (v_a, v_a) = 1$ and $(v_{a+1}, v_{a+1}) = \dots = (v_n, v_n) = -1$.

Equivalently, for any real invertible symmetric A there is a real X such that $XAX^T = \text{diag}[1, \dots, 1, -1, \dots, -1]$, and the number of 1's is independent of X .

18 Symplectic groups

If f is alternating and nondegenerate, pick any $0 \neq v \in V$. Then $v \notin V^\perp$ so $\exists w \in V : (v, w) = 1, (w, v) = -1$.

And $W = kv + kw$ is nondegenerate, so $V = W \oplus W^\perp$. On W , the form has matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Such a 2-dimensional subspace is called a **hyperbolic plane**.

Theorem (One alternating form only)

If f is nondegenerate and alternating, then V is the orthogonal sum of hyperbolic planes. In particular $\dim V$ is even.

The notations $Sp(V)$ and $Sp(2n, k)$ are common. When $k = \mathbb{R}$, $Sp(2n)$ and $S(2n)$ are used. $Sp(2n)$ is **not compact**.

Theorem

$Sp(V) \leq SL(V)$, i.e., $\det X = 1$ for all $X \in Sp(V)$. Moreover $Sp(V) \cap Z(V) = \langle \pm 1 \rangle$. If $\dim V = 2$, then $Sp(V) = SL(V)$.

19 Unitary groups

With respect to complex conjugation, for any positive definite hermitian symmetric form on $V = \mathbb{C}^n$, there is an orthonormal basis.

If $X \in U(n)$, then $X\bar{X}^T = I$, and so $\delta := \det X$ satisfies $\delta\bar{\delta} = 1$, so $\delta \in S^1$.

Like $O(n)$, $U(n)$ is compact.

20 A Spin Covering Example: $SU(2) \longrightarrow SO(3)$

Start with the real vector space

$$\mathfrak{u} := \left\{ \left[\begin{array}{cc} a & b + ci \\ b - ci & -a \end{array} \right] \mid a, b, c \in \mathbb{R} \right\} = \{X \mid \bar{X}^T = X \text{ and } \text{Tr}(X) = 0\}.$$

Then $\dim_{\mathbb{R}}(\mathfrak{u}) = 3$, and the form $(X, Y) = \text{Tr}(XY)/2$ is symmetric bilinear, with $(X, X) = a^2 + b^2 + c^2$.

Thus, $\mathfrak{u} \sim \mathbb{R}^3$ via $X \mapsto [a \ b \ c]$.

Now bring in $SU(2)$. If $X \in \mathfrak{u}$ and $U \in SU(2)$, write

$$c_U(X) = UXU^{-1} = UX\bar{U}^T \in \mathfrak{u} \quad !!! \quad \implies c_U \in GL(\mathfrak{u})$$

$$\text{Tr}((c_U X)^2) = \text{Tr}(c_U(X^2)) = \text{Tr}(X^2) \implies c_U \in O(\mathfrak{u}) \cong O(3)$$

$c_V(c_U(X)) = VXU^{-1}V^{-1} = c_{VU}(X)$, so π is a homomorphism:

$$\pi : SU(2) \longrightarrow O(\mathfrak{u}) \cong O(3), \quad U \mapsto c_U$$

One computes $\ker \pi = \langle \pm 1 \rangle$, and $\text{im } \pi = SO(\mathfrak{u})$.

Every orthogonal group has a “spin covering” $Spin(V) \rightarrow SO(V)$.

Here, $Spin(3) \cong SU(2)$.

21 More on simplicity

- Most classical groups have a simple “core” like $PSL(V)$, in between an abelian quotient and a central subgroup.

One key to the core’s simplicity is the geometric **homogeneity**.

Theorem (Witt’s Lemma)

Let V have a nondegenerate symmetric, alternating, or Hermitian-symmetric form. Let W and W' be isometric subspaces of V . Then any isometry $T : W \rightarrow W'$ extends to an isometry T^ of V .*

$$\begin{array}{ccc} W & \xrightarrow{T} & W' \\ \downarrow i & & \downarrow i \\ V & \xrightarrow{T^*} & V \end{array}$$

- Over an algebraically closed field, in addition to the classical groups – L , S , O – there are exactly 5 simple algebraic groups. Over \mathbb{C} the same is true for simple Lie groups.

Over a finite field F or $F = \mathbb{R}$, each of the classical groups over the algebraic closure \overline{F} or \mathbb{C} gives rise to one or more “ F -forms,” which are finite groups or real Lie groups (resp.) with a simple core as above, e.g. $SU(n)$ and $SL_n(\mathbb{R})$ are two forms of $SL_n(\mathbb{C})$.

THE END

22 Complements

- ▶ **p.2** $\mathbf{P}^2(\mathbb{R})$ can be obtained from the sphere S^2 (i.e., the surface of the Earth) by identifying each point with its antipode (opposite).
- ▶ **p.3** Each column of $M_{C,B}$ consists of the coordinates of one vector in C with respect to the basis B .
 $PGL(V)/PSL(V) \cong k^\times / (k^\times)^n$, where $(k^\times)^n$ is the multiplicative group of n^{th} powers of elements of k^\times .
 $Z(V) \cap SL(V) = Z(SL(V))$ is isomorphic to the group of n^{th} roots of unity in k .
- ▶ **p.3** One can define the groups GL , SL , PGL , etc., using a ring Y instead of a field k . The group structure is much more complicated in this case. Such groups (particularly $PSL_2(\mathbb{Z})$) are fundamental in number theory.
- ▶ **p.4** There are complements to $GL(V)$ in $\Gamma L(V)$. Fix a basis B of V . For any $\sigma \in \text{Aut}(k)$, $\sigma_B(\sum \alpha_i v_i) = \sum \sigma(\alpha_i) v_i$ defines $\sigma_B \in \Gamma L(V)$. Then $H_B := \{\sigma_B \mid \sigma \in \text{Aut}(k)\}$ is a complement to $GL(V)$.

23 Complements, c't'd

- ▶ **p.3** The **fractional linear transformation**

$$z \mapsto \frac{az + b}{cz + d}$$

is induced on $\mathcal{P}^1(k) = k \cup \{\infty\}$ by applying the element of $PGL_2(k)$ corresponding to $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(k)$.

- ▶ **p.3** In the case $n = 2$, the group $PGL(V)$ is **sharply triply transitive** on $\mathcal{P}(V) = \mathcal{P}^1(k)$: for any six points $P, Q, R, P', Q', R' \in \mathcal{P}(V)$ such that P, Q, R are distinct and P', Q', R' are distinct, there is a unique $g \in PGL(V)$ such that $gP = P'$, $gQ = Q'$, and $gR = R'$.
- ▶ **p.7** Simplicity when $n = 2$. The proof doesn't cover the case $n = 2$, $G \neq NQ$. But Since $GL_1(k) \cong k^\times$ is abelian, G/NQ is abelian. Repeat the final two steps with NQ in place of N to reach the contradiction $NQ = G$. The condition $|k| > 3$ is required for Q to be generated by commutators. Commutators of the form $[g, h]$, g diagonal, $h \in Q$, do the trick.

24 Complements, c't'd

- ▶ **p.8, 16** The group of all **rigid** (i.e., distance-preserving) **motions** of \mathbb{R}^n is the affine orthogonal group $AO(n)$.
- ▶ **p.9** In the study of quadratic forms over fields of characteristic 2 one often assumes that every element of the field is a square.
- ▶ **p.9** If $2 \neq 0$, then a symmetric form $(\ , \)$ gives rise to a quadratic form q by $q(v) = (v, v)$. If $2 \neq 0$, the quadratic form gives back the symmetric form as $(v, w) = \frac{1}{2}[q(v+w) - q(v) - q(w)]$, so q and $(\ , \)$ determine each other. If $2 = 0$, then $(\ , \)$ gives rise to q as before, but q is degenerate: $q(v+w) - q(v) - q(w) = 0$.
- ▶ **p.12** If $2 \neq 0$, and f and g are symmetric, then the isometry condition is equivalent to
$$g(Tv, Tv) = f(v, v) \text{ for all } v \in V.$$
- ▶ **p.13** A quadratic form q is called nondegenerate \iff its corresponding symmetric bilinear form is nondegenerate.

25 Complements, c't'd

- ▶ **p.13** X^\perp is always a subspace of V .
- ▶ **p.13** (2)–(5) are equivalent: $\dim V = \dim V^*$ (V is fin-dim), and $V^\perp = \ker \phi$. So (2) \iff (3) $\iff \phi$ is an isomorphism. Define $\rho_X : V^* \rightarrow X^*$ by $\rho_X(f) = f|_X$ (restriction to X); ρ_X is surjective. If ϕ is an isomorphism, then $\psi := \rho_X \circ \phi$ is surjective and $n = \dim \ker \psi + \dim X^* = \dim X^\perp + \dim X$. Therefore (2) \implies (4). By dimension count, (4) \implies (5). Finally take $X = 0$ in (5) to get (1).
- ▶ **p.15** The following theorem can be found in Artin's *Geometric Algebra*: Any nondegenerate bilinear form on V with the property that $(v, w) = 0 \implies (w, v) = 0$ must be either symmetric or alternating.
- ▶ **p.15** Only when there is no danger of confusion should f or A be omitted from the group notation.

