

# Multilinear Algebra and Tensor Symmetries

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$\mathbb{F}$  = field:  $\mathbb{R}, \mathbb{C}, \dots$

$V$  = finite-dimensional vector space over  $\mathbb{F}$  ( $\dim V = n$ )

$V \longleftrightarrow \mathbb{F}^n = n \times 1$  column vectors when basis  $\{\mathbf{v}_i\}$  fixed

$GL(V)$  = group of invertible linear transformations  $g : V \rightarrow V$   
 $\longleftrightarrow GL(n, \mathbb{F})$  ( $n \times n$  matrices with  $\det \neq 0$ )

$V^*$  = dual space of linear functions  $\mathbf{v}^* : V \rightarrow \mathbb{F}$  **covectors**

**Duality pairing**  $V^* \times V \rightarrow \mathbb{F}$  (bilinear):  $\langle \mathbf{v}^*, \mathbf{u} \rangle \stackrel{\text{def}}{=} \mathbf{v}^*(\mathbf{u})$

$V^* \longleftrightarrow 1 \times n$  row vectors using **dual basis**:  $\langle \mathbf{v}_j^*, \mathbf{v}_i \rangle = \delta_{ij}$

When basis/dual basis fixed, then  $\langle \mathbf{v}^*, \mathbf{u} \rangle = \mathbf{v}^* \mathbf{u}$ .

(row vector  $\times$  column vector = scalar)

If  $g \in GL(V)$ , then **transpose**  ${}^t g \in GL(V^*)$ :  $\langle {}^t g \mathbf{v}^*, \mathbf{u} \rangle \stackrel{\text{def}}{=} \langle \mathbf{v}^*, g \mathbf{u} \rangle$ .

Calculate as  $\mathbf{v}^* g$  (matrix product) when  $\mathbf{v}^*$  = row vector.

**Same as:** Use (transposed matrix)  $\times$  (column vector)

# Change of Basis vs. Moving a Vector

- Express  $\mathbf{u}$  in terms of basis  $\{\mathbf{v}_i\}$  or basis  $\{g\mathbf{v}_i\}$  ( $g \in GL(V)$ ).
- Change vector  $\mathbf{u}$  to  $g\mathbf{u}$  (**orbit** of  $\mathbf{u}$  under action of  $GL(V)$ ).

**Expansion Formulas:**  $\mathbf{u} = \sum_i \langle \mathbf{v}_i^*, \mathbf{u} \rangle \mathbf{v}_i$ ,  $\mathbf{v}^* = \sum_i \langle \mathbf{v}^*, \mathbf{v}_i \rangle \mathbf{v}_i^*$   
for  $\mathbf{u} \in V$  and  $\mathbf{v}^* \in V^*$

Components of  $\mathbf{u}$  relative to basis  $\{\mathbf{v}_i\}$  are  $x_i \stackrel{\text{def}}{=} \langle \mathbf{v}_i^*, \mathbf{u} \rangle$ .

Components of  $\mathbf{v}^*$  relative to basis  $\{\mathbf{v}_i^*\}$  are  $y^i \stackrel{\text{def}}{=} \langle \mathbf{v}^*, \mathbf{v}_i \rangle$ .

**Key Property:**  $\langle {}^t g^{-1} \mathbf{v}^*, g\mathbf{u} \rangle = \langle \mathbf{v}^*, g^{-1} g\mathbf{u} \rangle = \langle \mathbf{v}^*, \mathbf{u} \rangle$

Hence  $f(\mathbf{v}^*, \mathbf{u}) \stackrel{\text{def}}{=} \langle \mathbf{v}^*, \mathbf{u} \rangle$  is a  **$GL(V)$ -invariant function** on  $V^* \times V$ .

**Consequences:**

- $\{{}^t g^{-1} \mathbf{v}_i^*\}$  is the dual basis to  $\{g\mathbf{v}_i\}$
- Components of  $g\mathbf{u}$  relative to basis  $\{g\mathbf{v}_i\}$  are the same  $\{x_i\}$ .
- Components of  ${}^t g^{-1} \mathbf{v}^*$  relative to basis  $\{{}^t g^{-1} \mathbf{v}_i^*\}$  are the same  $\{y^i\}$ .
- $\langle \mathbf{v}^*, \mathbf{u} \rangle = \sum_i y_i x^i$  (**contraction of covector and vector**)

**Question:** How do we multiply vector spaces?

Let  $U, V$  be finite-dimensional vector spaces. Fix bases  $\{\mathbf{u}_i\}, \{\mathbf{v}_j\}$ . Define **tensor product**  $U \otimes V =$  vector space with basis  $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$

- $\mathbf{x} = \sum_{ij} x_{ij} \mathbf{u}_i \otimes \mathbf{v}_j \in U \otimes V$  has components  $\{x_{ij}\}$  relative to basis  $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$ .
- $\dim U \otimes V = \dim U \dim V$ .
- Use dual bases to get bilinear map  $\tau : U \times V \rightarrow U \otimes V$   

$$\tau(\mathbf{u}, \mathbf{v}) = \sum_{i,j} x_i y_j \mathbf{u}_i \otimes \mathbf{v}_j \stackrel{\text{def}}{=} \mathbf{u} \otimes \mathbf{v} \quad \text{Kronecker product}$$
 Here  $\{x_i\} =$  components of  $\mathbf{u}$ ,  $\{y_j\} =$  components of  $\mathbf{v}$

**Universal Linearization Property:** Let  $W$  be any vector space, and  $\beta : U \times V \rightarrow W$  a **bilinear map** (linear in each variable)

Set  $B(\sum_{i,j} x_{ij} \mathbf{u}_i \otimes \mathbf{v}_j) = \sum_{ij} x_{ij} \beta(\mathbf{u}_i, \mathbf{v}_j)$

- $B : U \otimes V \rightarrow W$  is linear
- $B(u \otimes v) = \beta(u, v)$

# Functoriality of Tensor Products

Let  $\text{Hom}(U, V) =$  all linear maps  $T : U \rightarrow V$

## Theorem

If  $X, Y$  are vector spaces and  $S \in \text{Hom}(U, X)$ ,  $T \in \text{Hom}(V, Y)$ , then there exists a unique  $S \otimes T \in \text{Hom}(U \otimes V, X \otimes Y)$  such that

$$(S \otimes T)(\mathbf{u} \otimes \mathbf{v}) = (S\mathbf{u}) \otimes (T\mathbf{v}) \quad \text{for } \mathbf{u} \in U, \mathbf{v} \in V.$$

This sets up a linear isomorphism

$$(\star) \quad \text{Hom}(U, X) \otimes \text{Hom}(V, Y) \cong \text{Hom}(U \otimes V, X \otimes Y)$$

(Each side has dimension =  $\dim U \dim X \dim V \dim Y$ .)

## Special Cases of $(\star)$

- $X = Y = \mathbb{F} \cong \mathbb{F} \otimes \mathbb{F}$ , so  $U^* \otimes V^* \cong (U \otimes V)^*$   
Basis  $\{\mathbf{u}_i^* \otimes \mathbf{v}_j^*\}$  for  $U^* \otimes V^*$  dual to basis  $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$  for  $U \otimes V$   
Components of  $\mathbf{x} \in U \otimes V$  are  $x_{ij} = \langle \mathbf{u}_i^* \otimes \mathbf{v}_j^*, \mathbf{x} \rangle$
- $U \otimes V^* \cong \text{Hom}(V, U)$ :  
 $\mathbf{u} \otimes \mathbf{v}^*$  gives transformation  $T_{\mathbf{u}, \mathbf{v}^*} : \mathbf{x} \mapsto \langle \mathbf{v}^*, \mathbf{x} \rangle \mathbf{u}$   
 $U = \mathbb{F}^m, V = \mathbb{F}^n$ :  $\text{Hom}(V, U) = m \times n$  matrices  
 $T_{\mathbf{u}, \mathbf{v}^*} = \mathbf{u}\mathbf{v}^*$  (column  $\times$  row) rank one matrix

# Iterated Tensor Products: Linearizing Multilinear Maps

**Associativity of Tensor Product:**  $U, V, W$  vector spaces

Define bilinear map  $\tau : (U \otimes V) \times W \rightarrow U \otimes (V \otimes W)$  by

$$\tau(\mathbf{u} \otimes \mathbf{v}, \mathbf{w}) = \mathbf{u} \otimes (\mathbf{v} \otimes \mathbf{w}).$$

Construction of tensor product gives isomorphism

$$T : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$$

On basis:  $T(\mathbf{u}_i \otimes \mathbf{v}_j) \otimes \mathbf{w}_k = \mathbf{u}_i \otimes (\mathbf{v}_j \otimes \mathbf{w}_k)$

Write  $(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W$  (note order)

$\mathbf{x} = \sum_{i,j,k} x_{ijk} \mathbf{u}_i \otimes \mathbf{v}_j \otimes \mathbf{w}_k$  components  $x_{ijk} = \langle \mathbf{u}_i^* \otimes \mathbf{v}_j^* \otimes \mathbf{w}_k^*, \mathbf{x} \rangle$

**Linearization Property:** If  $f : U \times V \times W \rightarrow Z$  is a trilinear map, then there exists unique **linear** map  $F : U \otimes V \otimes W \rightarrow Z$  with

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = F(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}).$$

**General Case:** For vector spaces  $V_1, \dots, V_p, Z$  the tensor product  $V_1 \otimes \dots \otimes V_p$  has basis  $\{\mathbf{v}_{i_1} \otimes \dots \otimes \mathbf{v}_{i_p}\}$  and linearizes  $p$ -multilinear maps  $f : V_1 \times \dots \times V_p \rightarrow Z$ .

**Notation:**  $V^{\otimes p} = V \otimes \cdots \otimes V$  ( $p$  factors)

$$V^{\otimes(p,q)} = V^{\otimes p} \otimes (V^*)^{\otimes q} \quad \text{mixed tensors of type } (p, q)$$

Basis for  $V^{\otimes(p,q)}$  from basis/dual basis for  $V$  and  $V^*$ :

$$\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p} \otimes \mathbf{v}_{k_1}^* \otimes \cdots \otimes \mathbf{v}_{k_q}^*\} \quad (i_j, k_j = 1, \dots, n = \dim V)$$

**Classic Tensor Notation:** Components of  $\mathbf{x} \in V^{\otimes(p,q)}$  are written

$$\mathbf{x}_{i_1 \cdots i_p}^{k_1 \cdots k_q} \stackrel{\text{def}}{=} \langle \mathbf{v}_{i_1}^* \otimes \cdots \otimes \mathbf{v}_{i_p}^* \otimes \mathbf{v}_{k_1} \otimes \cdots \otimes \mathbf{v}_{k_q}, \mathbf{x} \rangle$$

Identify  $(V^*)^* = V$  as usual;  $\{\mathbf{v}_i\}$  is dual basis to  $\{\mathbf{v}_i^*\}$ .

Call  $i_1, \dots, i_p$  the **covariant** indices of  $\mathbf{x}$  and

$k_1, \dots, k_q$  the **contravariant** indices of  $\mathbf{x}$ .

**Representation of  $GL(V)$ :**

Define a group homomorphism  $\rho : GL(V) \rightarrow GL(V^{\otimes(p,q)})$  by

$$\begin{aligned} \rho(g)(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_p \otimes \mathbf{y}_1^* \otimes \cdots \otimes \mathbf{y}_q^*) = \\ g\mathbf{x}_1 \otimes \cdots \otimes g\mathbf{x}_p \otimes {}^t g^{-1}\mathbf{y}_1^* \otimes \cdots \otimes {}^t g^{-1}\mathbf{y}_q^* \end{aligned}$$

(for any  $\mathbf{x}_i \in V$  and  $\mathbf{y}_j^* \in V^*$ ).

Need  ${}^t g^{-1}$  on  $\mathbf{y}_j^*$  to have  $\rho(gh) = \rho(g)\rho(h)$  for  $g, h \in GL(V)$ .

For each  $1 \leq r \leq p$  and  $1 \leq s \leq q$ , define a linear map

$$C_r^s : V^{\otimes(p,q)} \rightarrow V^{\otimes(p-1,q-1)} \quad (r, s) \text{ contraction}$$

by taking components of  $C_r^s \mathbf{x}$  as

$$\sum_{1 \leq j \leq n} \mathbf{x}_{i_1 \dots i_{r-1} j i_{r+1} \dots i_{p-1}}^{k_1 \dots k_{s-1} j k_{s+1} \dots k_{q-1}} \quad (\text{set } i_r = k_s = j \text{ and sum on } j).$$

## Theorem

*Contractions commute with the action of  $GL(V)$  on mixed tensors.*

**Examples:** 1)  $V^{\otimes(1,1)} = V \otimes V^* \cong \text{End}(V)$ , and

$C_1^1 : V^{\otimes(1,1)} \rightarrow V^{\otimes(0,0)} = \mathbb{F}$  by  $C_1^1 \mathbf{x} = \sum_j x_j^j = \text{tr}(\mathbf{x})$  (trace of  $\mathbf{x}$ )

If  $\mathbf{x} = \mathbf{u} \otimes \mathbf{u}^*$  then  $C_1^1(\mathbf{x}) = \langle \mathbf{u}^*, \mathbf{u} \rangle$ .

2) Take  $\mathbb{F} = \mathbb{R}$  and  $V =$  tangent space at a point of a Riemannian manifold. Then the **curvature tensor**  $R \in V^{\otimes(2,2)}$ .

contraction of  $R$  gives **Ricci curvature**  $\text{Ric} \in V^{\otimes(1,1)}$

contraction (trace) of  $\text{Ric}$  gives **scalar curvature** in  $V^{\otimes(0,0)} = \mathbb{R}$

“Contraction is an operation of almost magical efficiency”

(*Tensor Analysis*, Encyclopedia Britannica, 14th ed.)

$\mathfrak{S}_k$  = symmetric group (permutations of  $\{1, \dots, k\}$ )

Permutation  $s \in \mathfrak{S}_k$  acts on  $V^{\otimes k}$  by moving the vectors:

$$\sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \mathbf{u}_{s^{-1}(1)} \otimes \cdots \otimes \mathbf{u}_{s^{-1}(k)}$$

(vector in position  $i \rightarrow$  position  $s(i)$ )

- $\sigma : \mathfrak{S}_k \rightarrow \text{GL}(V^{\otimes k})$  is a **representation**:  
 $\sigma(st) = \sigma(s)\sigma(t)$  and  $\sigma_k(1) = I$ .
- The transformation  $\sigma(s)$  commutes with the transformation  $\rho(g)$  for all  $s \in \mathfrak{S}_k$  and  $g \in \text{GL}(V)$ .

## Theorem (Schur duality)

(1) Any linear transformation on  $V^{\otimes k}$  that commutes with  $\rho(g)$  for all  $g \in \text{GL}(V)$  is a linear combination of  $\{\sigma(s) : s \in \mathfrak{S}_k\}$ .

(2) Any linear transformation on  $V^{\otimes k}$  that commutes with  $\sigma(s)$  for all  $s \in \mathfrak{S}_k$  is a linear combination of  $\{\rho(g) : g \in \text{GL}(V)\}$ .

**Symmetrizer operator**  $\mathbf{Sym} : V^{\otimes k} \rightarrow V^{\otimes k}$  (assume  $\text{char}(\mathbb{F}) = 0$ )

$$\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)$$

- $\mathbf{Sym}^2 = \mathbf{Sym}$  ( projection operator)

*Proof:*

$$\begin{aligned} \mathbf{Sym}^2 &= \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \sigma(s)\sigma(t) = \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \sigma(st) \\ &= \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \sigma(r) = \mathbf{Sym} \end{aligned}$$

- $\sigma(t)\mathbf{Sym} = \mathbf{Sym}$  for all  $t \in \mathfrak{S}_k$

*Proof:*

$$\sigma(t)\mathbf{Sym} = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma(t)\sigma(s) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \sigma(r) = \mathbf{Sym}$$

Define  $S^k(V) = \mathbf{Sym}(V^{\otimes k})$  (symmetric  $k$ -tensors)

If  $\mathbf{x} \in S^k(V)$  then the components  $x_{i_1 \dots i_k}$  are symmetric in the indices (unchanged under any transposition of indices), and conversely.

## Theorem

Let  $f : V \times \cdots \times V \rightarrow W$  ( $k$  factors) be any  $k$ -multilinear map that is *symmetric* in its arguments.

There is a unique *linear* map  $F : S^k(V) \rightarrow W$  such that

$$F(\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)) = f(\mathbf{u}_1, \dots, \mathbf{u}_k).$$

*Proof:* There exists linear map  $F : V^{\otimes k} \rightarrow W$  with

$$F(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = f(\mathbf{u}_1, \dots, \mathbf{u}_k)$$

Then  $F \circ \mathbf{Sym} = F$  since  $f$  is symmetric, so  $F \circ (I - \mathbf{Sym}) = 0$ .

## Example

$(V^*)^{\otimes 2} \longleftrightarrow$  bilinear forms on  $V$

$$\mathbf{b} = \sum_{i,j} b^{ij} \mathbf{v}_i^* \otimes \mathbf{v}_j^* \longleftrightarrow B(\mathbf{x}, \mathbf{y}) = \sum_{ij} b^{ij} x_i y_j$$

(written as  $b^{ij} x_i y_j$  in *Einstein summation notation*)

Then  $\mathbf{b} \in S^2(V^*) \iff b^{ij} = b^{ji} \iff B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$

For  $s \in \mathfrak{S}_k$  define  $\text{sgn}(s) = (-1)^m$  if  $s$  is a product of  $m$  transpositions. Then  $\text{sgn}(s)\text{sgn}(t) = \text{sgn}(st)$ ,  $\text{sgn}(1) = 1$ .

**Alternation operator**  $\mathbf{Alt} : V^{\otimes k} \rightarrow V^{\otimes k}$

$$\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \text{sgn}(s) \sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)$$

**Properties:**

- $\mathbf{Alt}^2 = \mathbf{Alt}$  ( projection operator)

*Proof:*

$$\begin{aligned} \mathbf{Alt}^2 &= \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \text{sgn}(s) \text{sgn}(t) \sigma(s) \sigma(t) \\ &= \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \text{sgn}(st) \sigma(st) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \text{sgn}(r) \sigma(r) \\ &= \mathbf{Alt} \end{aligned}$$

- $\sigma(t) \mathbf{Alt} = \text{sgn}(t) \mathbf{Alt}$  for all  $t \in \mathfrak{S}_k$

*Proof:*

$$\begin{aligned} \sigma(t) \mathbf{Alt} &= \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \text{sgn}(s) \sigma(t) \sigma(s) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \text{sgn}(t)^{-1} \sigma(r) \\ &= \text{sgn}(t) \mathbf{Alt} \end{aligned}$$

# Universal Linearization Property of $\bigwedge^k(V)$

Define  $\bigwedge^k(V) = \mathbf{Alt}(V^{\otimes k})$  (*alternating k-tensors*)

If  $\mathbf{x} \in \bigwedge^k(V)$  then the components  $x_{i_1 \dots i_k}$  are skew-symmetric in the indices (change sign under any transposition), and conversely.

## Theorem

Let  $f : V \times \dots \times V \rightarrow W$  ( $k$  factors) be any  $k$ -multilinear map that is alternating in its arguments (*changes sign when two arguments are permuted*).

There is a unique linear map  $F : \bigwedge^k(V) \rightarrow W$  such that

$$F(\mathbf{Alt}(\mathbf{u}_1 \otimes \dots \otimes \mathbf{u}_k)) = f(\mathbf{u}_1, \dots, \mathbf{u}_k).$$

*Proof:* There exists linear map  $F : V^{\otimes k} \rightarrow W$  with

$$F(\mathbf{u}_1 \otimes \dots \otimes \mathbf{u}_k) = f(\mathbf{u}_1, \dots, \mathbf{u}_k)$$

Then  $F \circ \mathbf{Alt} = F$  since  $f$  is alternating, so  $F \circ (I - \mathbf{Alt}) = 0$ .

**Notation:**  $\mathbf{Alt}(\mathbf{u}_1 \otimes \dots \otimes \mathbf{u}_k) = \mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_k$  (*exterior product of vectors*)

# Comparisons between Symmetric and Alternating Tensors

$S^k(V)$  ( $\dim V = n$ )

- basis  $\{\mathbf{Sym}(\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_k})\}$  labelled by indices  $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$
- $\dim S^k(V) = \binom{n+k-1}{k} = \#\{k \text{ balls in } n \text{ boxes}\}$
- $S^k(V) \cong \mathcal{P}^k(V^*) =$  homogeneous polynomials of degree  $k$  on  $V^*$  (**nonzero space** for all  $k$ ):  
 $\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) \longleftrightarrow$  monomial  $\langle \mathbf{u}^*, \mathbf{u}_1 \rangle \cdots \langle \mathbf{u}^*, \mathbf{u}_k \rangle$
- $\mathfrak{S}_k$  acts by identity on  $S^k(V)$
- $S^k(V)$  is invariant under  $GL(V)$ , and contains no proper subspace that is invariant under  $GL(V)$  (**irreducible representation** of  $GL(V)$ )
- **Quantum mechanics:**  $S^k(V)$  describes systems of  $k$  bosons

## $\wedge^k(V)$ ( $\dim V = n$ )

- basis  $\{\mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_k}\}$  labelled by indices  $1 \leq i_1 < i_2 < \cdots < i_k \leq n$
- $\dim \wedge^k(V) = \binom{n}{k} = \#\{\text{subsets of size } k \text{ in } n\text{-element set}\}$
- $\wedge^k(V) \cong$  homogeneous alternating functions of degree  $k$  on  $V^*$  (**zero** if  $k > n$ )
- $\mathfrak{S}_k$  acts by  $\text{sgn}$  on  $\wedge^k(V)$
- $\wedge^k(V)$  is invariant under  $\text{GL}(V)$ , and contains no proper subspace that is invariant under  $\text{GL}(V)$  (**irreducible representation** of  $\text{GL}(V)$ )
- $\dim \wedge^n V = 1$ . Fix basis vector  $\mathbf{u}$ . Then  $\rho(g)\mathbf{u} = \det(g)\mathbf{u}$  for  $g \in \text{GL}(V)$ .
- **Geometry:**  $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \neq 0 \iff \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  are linearly independent (span a  $k$ -plane).
- **Quantum Mechanics:**  $\wedge^k(V)$  describes systems of  $k$  *fermions* (Pauli exclusion principle)

The group  $GL(V) \times \mathfrak{S}_k$  acts on  $V^{\otimes k}$ .

**Program:** Decompose  $V^{\otimes k}$  into a direct sum of subspaces that are irreducible under the action of  $GL(V) \times \mathfrak{S}_k$

We already have two such spaces, namely  $S^k(V)$  and  $\Lambda^k(V)$ .

**Example** When  $k = 2$ , then  $V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V)$   
(Every bilinear form is the sum of a symmetric form and a skew-symmetric form)

For  $k > 2$ ?  $\dim V^{\otimes k} = n^k \gg \dim S^2(V) + \dim \Lambda^2(V)$

As a module for  $GL(V) \times \mathfrak{S}_k$ ,  $V^{\otimes k} \cong \bigoplus_{\lambda} E_{\lambda} \otimes F_{\lambda}$

- $\lambda$  runs over all *partitions* of  $k$  with at most  $\dim V$  parts
- $E_{\lambda}$  is an irreducible representation of  $GL(V)$ , and only occurs in the decomposition paired with  $F_{\lambda}$
- $F_{\lambda}$  is an irreducible representation of  $\mathfrak{S}_k$ , and only occurs in the decomposition paired with  $E_{\lambda}$ 
  - trivial representation of  $\mathfrak{S}_k \longleftrightarrow S^k(V)$
  - sgn representation of  $\mathfrak{S}_k \longleftrightarrow \bigwedge^k(V)$
- $GL(V)$  acts on the first tensor factor in  $E_{\lambda} \otimes F_{\lambda}$
- $\mathfrak{S}_k$  acts on the second tensor factor in  $E_{\lambda} \otimes F_{\lambda}$
- There are explicit operators (*Young symmetrizers*) that project on the subspaces  $E_{\lambda}$  and  $E_{\lambda} \otimes F_{\lambda}$

Lots of interesting algebra, analysis, and combinatorics!