Metric Spaces and Elementary Topology

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Introduction to Mathematics at Rutgers

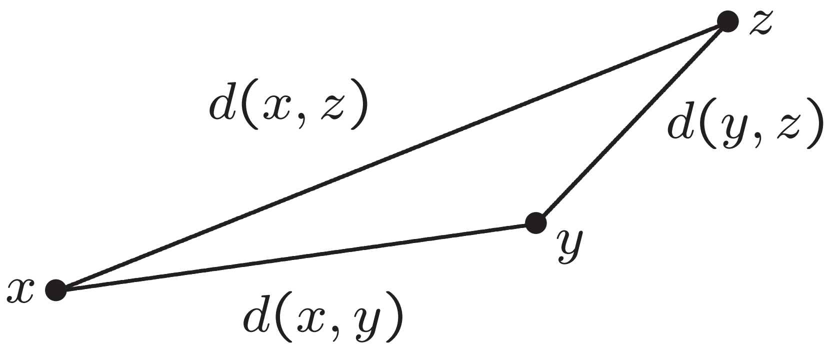
September 2006

I. BASICS

A. Definition

A *metric space* is a pair (X, d), where

- X is a set;
- $d : X \times X \to \mathbb{R}$ is a *metric* on X; $d(x, y)$ is interpreted as the distance between the points $x, y \in X$, and must satisfy
  1. $d(x, y) \geq 0$ for $x, y \in X$, with $d(x, y) = 0$ if and only if $x = y$;
  2. $d(x, y) = d(y, x)$ for $x, y \in X$ (symmetry);
  3. $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y, z \in X$ (the triangle inequality).



We usually write X rather than (X, d) when no confusion can arise.

B. Examples

- The Euclidean space $\mathbb{R}^n$: If $x, y \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, then the *Euclidean distance* between x and y is

$$d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

If we define the *length* of a vector $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ to be $\|z\| = \left(\sum_i z_i^2\right)^{1/2}$, then $d_2(x,y) = \|x - y\|_2$.

- A normed vector space V: If V is a real vector space then a *norm* $\|\cdot\|$ on V is a mapping from V to $\mathbb{R}$ such that

1. $\|x\| \geq 0$, with $\|x\| = 0$ iff $x = 0$;
2. $\|c\,x\| = |c|\|x\|$, $x \in V$, $c \in \mathbb{R}$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

Then $d(x,y) = \|x - y\|$ is a metric on V.

- $C([0,1])$: We let $C([0,1]) \equiv C$ denote the space of all continuous functions $f : [0,1] \to \mathbb{R}$, and define the *uniform norm* $\|\cdot\|_\infty$ on $C([0,1])$ by

$$\|f\|_\infty = \max_{t\in[0,1]} |f(t)|.$$

Thus C becomes a metric space:

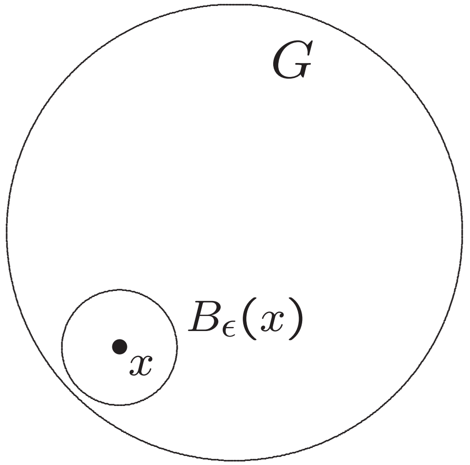
$$d(f,g) = \max_{t\in[0,1]} |f(t) - g(t)|.$$

Intuitively, two continuous functions in C are close if they are close at all points $x \in [0,1]$, i.e., *uniformly close*.

- Subspaces: If X is a metric space and Y an arbitrary subset of X then Y is also a metric space, with the metric inherited from X.

II. TOPOLOGY

A. Open sets



- For $\epsilon > 0$ and $x \in X$ the set

$$B_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$$

is called the *ϵ-ball* centered at x.

- A set $G \subset X$ is *open* if for every $x \in X$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset G$.

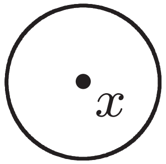
- The family of open sets has the properties:
  1. The empty set $\emptyset$, and X itself, are open;
  2. The union of an arbitrary family of open sets is open;
  3. The intersection of a finite number of open sets is open.

- Such a family of open sets defines a *topology*, as we will see below. Thus a metric gives a topology, but different metrics can give the same topology.

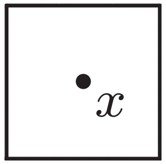
- Example: Equivalent metrics on $\mathbb{R}^n$: We may define other metrics on $\mathbb{R}^n$, e.g., those obtained from the norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \text{and} \qquad \|x\|_\infty = \max_{1 \le i \le n} |x_i|,$$

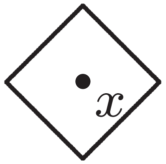
All these are equivalent in the sense that they define the same open sets—i.e., the same topology, although the balls they define look quite different: in $\mathbb{R}^2$ we have



d_2



d_∞



d_1

- We may also use the metric, again equivalent to d_2:

$$d(x,y) = \frac{d_2(x,y)}{1 + d_2(x,y)},$$

which does not come (directly) from a norm. Notice that in the last example the distance between two points is always less than 1. The balls defined with d are the same as those defined with d_2, and hence the topologies are certainly the same.

B. Topology in general—look for the yellow boxes

- If X is a set and $\mathcal{T}$ a collection of subsets of X (called *open sets*) with the above three properties:
  1. The empty set $\emptyset$, and X itself, are open;
  2. The union of an arbitrary family of open sets is open;
  3. The intersection of a finite number of open sets is open;

  then $\mathcal{T}$ is called a *topology* for X and $(X, \mathcal{T})$ (or just X) is a *topological space*.

- For later reference we mention here two special cases:
  - A topological space X is *Hausdorff* if for $x, y \in X$, with $x \neq y$, there are disjoint open sets $G_x, G_y \in \mathcal{T}$ with $x \in G_x$ and $y \in G_y$.
  - A space X is *first countable* if for each $x \in X$ there is a family $(G_{x,n})_{n=1}^{\infty}$ of open sets, all containing x, such that if an open set G contains x then $G_{x,n} \subset G$ for some n.

- A metric space is Hausdorff (if $d(x, y) = a$ take $G_x = B_{a/2}(x)$, $G_y = B_{a/2}(y)$) and first countable (take $G_{x,n} = B_{1/n}(x)$).

C. Closed sets and closure

- **Limit points:** A point x of X is a *limit point* of a set $S \subset X$ if every open set G containing x intersects S in at least one point other than x itself. A limit point of S may or may not itself belong to S.

- **Closed sets:** A set $F \subset X$ is *closed* if its complement $X \setminus F$ is open. Equivalently, F is closed if it contains all its limit points.

  From the basic properties of open sets, we see immediately that the empty set and X itself are closed, that an arbitrary intersection of closed sets is closed, and that a finite union of closed sets is closed.

- **Closure:** If S is a set in a topological space X, the *closure* of S, $\bar{S}$, is the smallest closed set containing S, or equivalently:

  The intersection of all closed sets containing S;

  The union of S and all its limit points.

D. Sequences

- A sequence $(x_n)_{n=1}^{\infty}$ of points in a topological space X *converges* to a point $x \in X$ if for every open set G containing x there is an $N \in \mathbb{N}$ such that $x_n \in G$; in this case we write $\lim_{n\to\infty} x_n = x$ and say that (x_n) is *convergent*.

- If X is a metric space, $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty} d(x_n, x) = 0$.

- If X is first countable—in particular, a metric space—then many topological concepts can be expressed in terms of sequences:
  - A point $x \in X$ is a limit point of the set $S \subset X$ iff there is a sequence (x_n) of distinct points with $x_n \in S$ and $\lim_{n\to\infty} x_n = x$.
  - A set $F \subset X$ is closed iff whenever (x_n) is a convergent sequence of points from F, $\lim_{n\to\infty} x_n \in F$.

III. Continuity

- Continuity in metric spaces: If X and Y are metric spaces, then $f : X \to Y$ is *continuous* if $f(x)$ and $f(y)$ are close together whenever x and y are. More precisely, f is *continuous* if and only if for every $x \in X$ and every $\epsilon > 0$ there is a $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

- Continuity via sequences: If X and Y are topological spaces, with X first countable—in particular, a metric space—then $f : X \to Y$ is continuous iff for every $x \in X$, $\lim_{n\to\infty} f(x_n) = f(x)$ whenever $\lim_{n\to\infty} x_n = x$.

- Continuity in general: Suppose that X and Y are topological spaces. Then a function $f : X \to Y$ is *continuous* if the inverse image $f^{-1}(G)$ of every open set $G \subset Y$ is an open subset of X.

- Examples: We are all familiar with continuous and discontinuous functions from $\mathbb{R}$ to $\mathbb{R}$. Somewhat more interesting are functions defined on C.

- Indefinite integration: Define $T : C \to C$ by $T(f)(x) = \int_0^x f(t)\,dt$. One checks that $T(f)$ is well defined and lies in C. Then T is itself continuous, since if $\|f - g\|_\infty < \epsilon$ then

$$\begin{aligned} |T(f)(x) - T(g)(x)| &= \left| \int_0^x [f(t) - g(t)]\,dt \right| \\ &\le \int_0^x |f(t) - g(t)|\,dt < \int_0^x \epsilon\,dx \le \epsilon \end{aligned}$$

and hence $\|T(f) - T(g)\| < \epsilon$.

- Differentiation: Let $C^1 \subset C$ denote the set of functions in C which have continuous (first) derivatives; C^1 is a metric space (with the subspace topology) and we can define the operator $D : C^1 \to C$ by $D(f) = f'$. But D is not continuous, since the sequence (f_n) defined by $f_n = n^{-1} \sin \pi n x \in C^1$ satisfies $\|f_n\|_\infty = n^{-1}$ and hence $\lim_{n\to\infty} f_n = 0$, but $Df_n = \pi \cos n\pi x$ satisfies $\|Df_n\|_\infty = \pi$.

IV. COMPLETENESS

A. Definition

- **Cauchy sequences:** Suppose that X is a metric space. A sequence (x_n) of points of X is a *Cauchy sequence* if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

which means that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

- **Completeness:** A metric space X is *complete* if every Cauchy sequence (x_n) in X is convergent: $\lim_{n\to\infty} x_n = x$ for some $x \in X$.

- **Example:** The set $\mathbb{Q}$ of rational numbers, with the usual metric (inherited from $\mathbb{R}$), is not complete; for example, if $x_n = max\{k/n \mid k \in \mathbb{N}, (k/n)^2 < 2\}$ then (x_n) is a Cauchy sequence with no limit in $\mathbb{Q}$. On the other hand, the set $\mathbb{R}$ of real numbers is complete; essentially because the $\mathbb{R}$ has the *least upper bound property*: every set of real numbers which is bounded above has a least upper bound. $\mathbb{R}^n$ is also complete.

- **Example:** The space $C = C([0,1])$ is complete. To see this, let (f_n) be a Cauchy sequence of functions in C. Then for any fixed $x \in [0,1]$, $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, and hence convergent; this means that there is a function $f : [0,1] \to \mathbb{R}$ with $\lim_{x\to\infty} f_n(x) = f(x)$ for every $x \in [0,1]$. The fact that $\lim_{m,n\to\infty} \|f_n - f_m\|_\infty = 0$ easily implies that also $\lim_{n\to\infty} \|f_n - f\| = 0$, and the result follows from the fact that the uniform limit of continuous functions is continuous.

- **Completion:** If (X, d_X) is a metric space then one may construct a complete metric space (Y, d_Y) such that $X \subset Y$, the restriction of the metric d_Y to X is d_X and the closure of X in Y. Y is (essentially) uniquely determined and is called the *completion* of X.

- For example, the completion of $\mathbb{Q}$ is $\mathbb{R}$.

B. The Contraction Mapping Principle

Let (X, d) be a metric space. A mapping $f : X \to X$ is a *contraction* if there exists a number α, with $0 \leq \alpha < 1$, such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha\, d(x, y) \qquad (*)$$

It is easy to see that a contraction must be continuous.

A *fixed point* of f is a point $x \in X$ such that $f(x) = x$.

Theorem: *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point x in X.*

Proof: Choose $\alpha < 1$ so that f satisfies $(*)$. Let x_0 be any point of X and inductively define $x_n = f(x_{n-1})$ for $n = 1, 2, \ldots$. We claim that for any $n \geq 0$, $d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1)$; this is trivial for $n = 0$ and follows by induction for general n:

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \alpha\, d(x_{n-1}, x_n) \leq \alpha^n d(x_0, x_1)$$

Proof (continued): But then if $m \geq n \geq 0$, then the triangle inequality yields

$$\begin{aligned} d(x^n, x^m) &\leq \sum_{i=0}^{m-n+1} d(x_{n+i}, x_{n+i+1}) \\ &\leq \sum_{i=0}^{m-n+1} \alpha^{n+i} d(x_0, x_1) \;\leq\; \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \end{aligned}$$

so that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and (x_n) is Cauchy. Let $x = \lim_{n\to\infty} x_n$; then from the continuity of f, x is a fixed point of f:

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

If y is any fixed point then $d(x, y) = d(f(x), f(y)) \leq \alpha\, d(x, y)$, which is possible only if $d(x, y) = 0$, i.e., $x = y$. ■

The contraction mapping principle is often applied in the following form.

Corollary *Suppose that X is a complete metric space, $B \subset X$ a ball, and $f : X \to X$ a mapping such that $f(B) \subset B$ and f is a contraction when restricted to B. Then f has a fixed point in B.*

V. COMPACTNESS

A. Basics

- **Definition:** Suppose that X is a topological space and K is a subset of X. A family $\{G_\alpha \mid \alpha \in A\}$ of open sets in X is an *open cover* of K if $K \subset \bigcup_{\alpha \in A} G_\alpha$. K is *compact* if every open cover of K contains a finite subcover, i.e., if for some $\alpha_1, \ldots, \alpha_n \in A$, $K \subset \bigcup_{1 \leq i \leq n} G_{\alpha_i}$.
- A compact subset of a Hausdorff space is closed.

- A subspace K of a metric space X is compact iff it is *sequentially compact*: every sequence (x_n) of points in K contains a convergent subsequence.

- A subset of $\mathbb{R}^n$ is compact iff it is *closed* and is *bounded* in the usual metric d_2. (But this is not true in the "equivalent" metric d defined by $d(x,y) = \dfrac{d_2(x,y)}{1 + d_2(x,y)}$, in which all sets are bounded.)

B. Functions on a compact set

- Suppose that K is compact and $f : K \to \mathbb{R}$ is continuous. Then:
  - f is *bounded:* for some $M > 0$, $|f(x)| \leq M$ for all $x \in K$.
  - f has a *maximum* and *minimum* on K: for some $x_1, x_2 \in K$, $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in K$.

- If K is a compact metric space and f is as above then:
  - f is *uniformly continuous:* for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ if $x, y \in X$ with $d(x, y) < \delta$.

- Proof of boundedness: Let $S_n = (-n, n) \subset \mathbb{R}$. Each S_n is open, so that the sets $G_n = f^{-1}(S_n)$ form an open cover of K, which must have a finite subcover. Since $G_n \subset G_m$ if $n < m$, $K \subset G_N$ for some N, i.e., $|f(x)| < N$ for all $x \in K$.

- If K is a compact metric space and (f_n) is a sequence of functions with $f_n : K \to \mathbb{R}$ which is
  - *uniformly bounded*: for some $M > 0$ and all $n \in \mathbb{N}$ and $x \in K$, $|f_n(x)| \leq K$; and
  - *uniformly equicontinuous*: for every and every $\epsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$, $|f_n(x) - f_n(y)| < \epsilon$ if $d(x, y) < \delta$;

  then (f_n) contains a subsequence converging uniformly to some continuous function $f : K \to \mathbb{R}$.

- Example: Given $M, M_1 \geq 0$ let $S \subset C([0, 1])$ be the set of continuously differentiable functions f such that $|f| \leq M$ and $|f'| \leq M_1$. Then the closure $\bar{S}$ is compact.

- Proof sketch: Every sequence (f_n) of functions in S is uniformly bounded, since $|f_n(x)| \leq M$ for all n and x, and uniformly equicontinuous, if $1 \leq x < y \leq 1$ then from the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(z)|\,|x - y| \leq M_1|x - y|.$$

for some z with $x < z < y$. Thus (f_n) has a convergent subsequence; the limit belongs to $\bar{S}$ and it follows easily that $\bar{S}$ is sequentially compact and hence compact. ■

VI. CONNECTEDNESS

- **Definition:** A subset S of a topological space X is *connected* if there do not exist two open subsets of X, G_1 and G_2, such that $S \subset G_1 \cup G_2$ and $S \cap G_1$ and $S \cap G_2$ are disjoint and are both not empty.

- **Example:** Suppose that S is a subset of $\mathbb{R}$ that contains both positive and negative numbers, but not zero. Then S is not connected, as shown by consideration of the open subsets $G_1 = (0, \infty)$ and $G_2 = (-\infty, 0)$ of $\mathbb{R}$.

By the same reasoning we see that if a connected subset $S \subset \mathbb{R}$ contains distinct points x and y then it must also contain all intermediate points z and conclude that the only connected subsets of $\mathbb{R}$ are the intervals - open, closed, half open, semi or doubly infinite, etc.

- **Definition:** A subset S of a topological space X is *path connected* if for each $x, y \in S$ there is a *path* from x to y in S: a continuous map $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

- **Example:** The *topologist's sine curve* $S \subset \mathbb{R}^2$,

$$S = \{(x,y) \in \mathbb{R}^2 \mid x = 0,\ -1 \le y \le 1\} \cup \left\{ \left(x, \sin \frac{1}{x} \right) \in \mathbb{R}^2 \mid x > 0 \right\},$$

is connected but not path connected.