# Integral Transforms

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$G$ – locally compact abelian topological group
$G$ has a Haar measure (translation invariant)
$L^2(G)$ – square integrable complex valued functions on $G$
Inner product: $(\phi, \psi) = \int_G \phi(x)\overline{\psi(x)}\,\mathrm{d}x$
norm: $\|\phi\|_2 = \sqrt{(\phi, \phi)}$
$G$ acts on $L^2(G)$ by translations: $T_y\phi(x) = \phi(x - y)$

### Definition

A linear transformation (operator) $C : L^2(G) \to L^2(G)$ is translation invariant if it commutes with $\{T_y\}_{y \in G}$.

Some Examples

- Translation: $C\phi = T_y\phi$ with $y \in G$
- Convolution: $C\phi(x) = \int_G f(y)\phi(x - y)\,dy$ with $f \in L^1(G)$ (weighted average of translates of $\phi$)

Problem: Diagonalize all translation invariant operators
Solution: Use characters of $G$ and Fourier transform

Example 1 $\quad G = \mathbb{Z}/n\mathbb{Z}$ (additive group of integers mod $n$)

$L^2(G) = \{\phi : \mathbb{Z} \to \mathbb{C} \ : \ \phi(k+n) = \phi(k) \text{ for all } k \in \mathbb{Z}\}$

inner product $(\phi, \psi) = \frac{1}{n} \sum_{k=0}^{n-1} \phi(k)\overline{\psi(k)}$

Characters $\quad e_p(k) = w^{kp}$ for $k, p \in \mathbb{Z} \quad (w = \mathrm{e}^{2\pi\mathrm{i}/n}, \ \ w^n = 1)$

- $e_p(k+m) = e_p(k)e_p(m), \quad |e_p(k)| = 1, \quad e_{p+n} = e_p$
- Eigenfunctions for translations $\quad T_k e_p = w^{-kp} e_p$
- Orthogonality relations

$$(e_p, e_q) = \begin{cases} 1 & \text{if } p - q \equiv 0 \mod (n) \\ 0 & \text{else} \end{cases}$$

Finite Fourier Transform $\quad \widehat{\phi}(p) = (\phi, e_p)$

- Diagonalization $\quad \psi = T_k \phi \quad \Rightarrow \quad \widehat{\psi}(p) = w^{-kp}\widehat{\phi}(p)$
- Fourier inversion $\quad \phi = \sum_{p=0}^{n-1} \widehat{\phi}(p) e_p$
- Plancherel formula $\quad (\phi, \psi) = \sum_{p=0}^{n-1} \widehat{\phi}(p)\overline{\widehat{\psi}(p)}$

**Theorem**

*Let $G = \mathbb{Z}/n\mathbb{Z}$. Let $C$ be a translation invariant operator on $L^2(G)$. There is a function $F$ on $\widehat{G} \cong \mathbb{Z}/n\mathbb{Z}$ such that*

$(\star) \qquad \widehat{C\phi}(p) = F(p)\widehat{\phi}(p)$ *for all* $\phi \in L^2(G)$ *and* $p \in \mathbb{Z}$.

*Conversely, every function $F$ on $\mathbb{Z}/n\mathbb{Z}$ defines a translation invariant operator $C$ on $L^2(G)$ by $(\star)$ ($C =$ convolution by $f$, where $\widehat{f} = F$).*

**Proof.**

Let $S = T_1$ (shift operator). Then $S$ has $n$ distinct eigenvalues $\lambda_p = w^{-p}$ for $p = 0, \ldots, n-1$ with eigenvectors $e_p$. Since $C$ commutes with $S$, the function $Ce_p$ is an eigenvector for $S$ with eigenvalue $w^{-p}$. Hence $Ce_p = F(p)e_p$ for some scalar $F(p) \in \mathbb{C}$. The Fourier inversion formula now implies $(\star)$. $\square$

General Version of Fourier Transform

$G$ – locally compact abelian topological group (written additively)
$\widehat{G}$ – all characters of $G$:

$\psi : G \to \mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ (continuous)
$\psi(x+y) = \psi(x)\psi(y), \quad \psi(0) = 1$

- $\widehat{G}$ is a locally compact abelian group under pointwise multiplication and uniform convergence on compacta topology.
- $\widehat{(\widehat{G})} \cong G$ (natural isomorphism, as for vector space duality).
- Fourier transform takes $L^2(G)$ onto $L^2(\widehat{G})$ preserving norm.
- Translation invariant operator $C$ on $L^2(G)$ becomes multiplication by a function $F$ on $L^2(\widehat{G})$.

Example

$G = \mathbb{Z}/n\mathbb{Z} \qquad \widehat{G} = \{e_p \ : \ p \in \mathbb{Z}/n\mathbb{Z}\} \cong G$
Choose basic character $e_1$. Then $e_p(k) = e_1(pk)$

Example 2 $G = \mathbb{R}/\mathbb{Z}$ (additive group of real numbers modulo 1)

$L^2(\mathbb{R}/\mathbb{Z})$ $\phi : \mathbb{R} \to \mathbb{C}$, $\phi(x+1) = \phi(x)$ (periodic, measurable)

$\int_0^1 |\phi(x)|^2 \, \mathrm{d}x < \infty$ (Lebesgue integral)

Inner product $(\phi, \psi) = \int_0^1 \phi(x)\overline{\psi(x)} \, \mathrm{d}x$

Characters $e_p(x) = \exp(2\pi \mathrm{i} p x)$ for $p \in \mathbb{Z}$ and $x \in \mathbb{R}$

- $e_p(x+1) = e_p(x)$, $|e_p(x)| = 1$
- $e_p(x)e_q(x) = e_{p+q}(x)$, so $\widehat{G} \cong \mathbb{Z}$ under $e_p \leftrightarrow p$
- Orthogonality relations $(e_p, e_q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{else} \end{cases}$

Fourier transform $\widehat{\phi}(p) = (\phi, e_p) = \int_0^1 \phi(x) \exp(-2\pi \mathrm{i} p x) \, \mathrm{d}x$

- Diagonalization $\psi = T_y \phi \Rightarrow \widehat{\psi}(p) = e_p(-y)\widehat{\phi}(p)$
- Fourier inversion $\phi = \sum_{p \in \mathbb{Z}} \widehat{\phi}(p) e_p$ ($L^2$ convergence)
- Plancherel formula $(\phi, \psi) = \sum_{p \in \mathbb{Z}} \widehat{\phi}(p)\overline{\widehat{\psi}(p)}$

# Bounded Translation Invariant Operators

Linear operator $C$ on $L^2(\mathbb{R}/\mathbb{Z})$ is bounded if $\|C\phi\|_2 \leq M\|\phi\|_2$
Same as: $C$ is a continuous transformation w.r.t. $\|\phi\|_2$.

## Theorem

*Let $C$ be a bounded translation invariant operator on $L^2(\mathbb{R}/\mathbb{Z})$. Then there is a bounded function $F$ on $\mathbb{Z}$ such that*

$(\star)$ $\qquad \widehat{C\phi}(p) = F(p)\widehat{\phi}(p)$ *for all* $\phi \in L^2(\mathbb{R}/\mathbb{Z})$.

*Conversely, every bounded function $F$ on $\mathbb{Z}$ defines a bounded translation invariant operator $C$ on $L^2(\mathbb{R}/\mathbb{Z})$ by $(\star)$.*

## Proof.

Let $S = T_y$, $y$ irrational. Then $S$ has distinct eigenvalues $\lambda_p = \exp(-2\pi i y p)$ for $p \in \mathbb{Z}$ with eigenvectors $e_p$. Hence $CS = SC \Rightarrow Ce_p = F(p)e_p$ with $F(p) \in \mathbb{C}$. Then $C$ bounded $\Rightarrow$ $\|F\|_\infty := \sup_p |F(p)| < \infty$. Hence

$$C\phi = \sum_{p\in\mathbb{Z}} \widehat{\phi}(p)Ce_p = \sum_{p\in\mathbb{Z}} \widehat{\phi}(p)F(p)e_p \qquad \square$$

$C^\infty(\mathbb{R}/\mathbb{Z}) =$ differentiable periodic functions on $\mathbb{R}$
$D = \frac{1}{2\pi \mathrm{i}} \frac{d}{dx}$ translation invariant operator on $C^\infty(\mathbb{R}/\mathbb{Z})$

- $De_p = pe_p$ for $p \in \mathbb{Z}$, so $D$ is not bounded on $L^2(\mathbb{R}/\mathbb{Z})$
- $(D\phi, \psi) = (\phi, D\psi)$ for $\phi, \psi \in C^\infty(\mathbb{R}/\mathbb{Z})$ (integrate by parts)
- $\widehat{D\phi}(p) = p\widehat{\phi}(p)$ for $\phi \in C^\infty(\mathbb{R}/\mathbb{Z})$
- $\phi \in C^\infty(\mathbb{R}/\mathbb{Z}) \iff \widehat{\phi}$ is rapidly decreasing:

  For every positive integer $r$ $\quad \sup_{p\in\mathbb{Z}} |p^r \widehat{\phi}(p)| < \infty$

### Theorem

*Let $C$ be a continuous translation invariant operator on $C^\infty(\mathbb{R}/\mathbb{Z})$. Then there is a function $F$ on $\mathbb{Z}$ of polynomial growth at $\infty$ such that*

$(\star) \qquad \widehat{C\phi}(p) = F(p)\widehat{\phi}(p)$ *for all* $\phi \in C^\infty(\mathbb{R}/\mathbb{Z})$.

*Conversely, every such function $F$ on $\mathbb{Z}$ defines a continuous translation invariant operator $C$ on $C^\infty(\mathbb{R}/\mathbb{Z})$ by $(\star)$.*

Example 3 $\quad G = \mathbb{R}$ (additive group of real numbers)

$L^2(\mathbb{R}) \quad \phi : \mathbb{R} \to \mathbb{C}$, (measurable) $\int_{-\infty}^{\infty} |\phi(x)|^2 \, \mathrm{d}x < \infty$

Inner product $\quad (\phi, \psi) = \int_{-\infty}^{\infty} \phi(x)\overline{\psi(x)} \, \mathrm{d}x$

Characters $\quad e_\xi(x) = \exp(2\pi \mathrm{i} x\xi)$ for $x, \xi \in \mathbb{R}$.

- Fix basic character $e_1$. Then $e_\xi(x) = e_1(x\xi)$
- $e_\xi(x)e_\tau(x) = e_{\xi+\tau}(x)$, so $\widehat{\mathbb{R}} \cong \mathbb{R}$ under $e_\xi \leftrightarrow \xi$
- $\mathbb{R}$ not compact $\Rightarrow e_\xi \notin L^2(\mathbb{R})$ (plane wave, frequency $\xi$)

Fourier transform For $\phi \in L^1(\mathbb{R})$ define

$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e_{-\xi}(x) \, \mathrm{d}x$ (integral converges absolutely)

- Fourier transform extends to isometry $L^2(\mathbb{R}) \to L^2(\widehat{\mathbb{R}})$
- Plancherel formula $\quad (\phi, \psi) = \int_{-\infty}^{\infty} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} \, \mathrm{d}\xi$
- Bounded translation invariant operator $C$ on $L^2(\mathbb{R}) \longleftrightarrow$ multiplication by bounded measurable function $F$ on $\widehat{\mathbb{R}}$

$\mathcal{S}(\mathbb{R}) =$ rapidly decreasing $C^\infty$ functions on $\mathbb{R}$:

$$\sup_{x\in\mathbb{R}} \left| x^m \left(\frac{d}{dx}\right)^k \phi(x) \right| < \infty \text{ for all positive integers } m, k$$

Example $\quad \phi(x) = p(x)e^{-\pi x^2}$ with $p(x)$ a polynomial

Fourier transform of $\phi$ is $q(\xi)e^{-\pi\xi^2}$ with $q$ a polynomial

- $\mathcal{S}(\mathbb{R})$ invariant under $D_x = \frac{1}{2\pi \mathrm{i}}\frac{d}{dx}$, $\; M_x =$ multiplication by $x$
- $\widehat{D_x\phi} = M_\xi\widehat{\phi}$ for $\phi \in \mathcal{S}(G)$ (integrate by parts)
- $\widehat{M_x\phi} = D_\xi\widehat{\phi}$ for $\phi \in \mathcal{S}(G)$ (differentiate under integral)
- $\phi \in \mathcal{S}(\mathbb{R}) \iff \widehat{\phi} \in \mathcal{S}(\mathbb{R})$

## Theorem

*Let $C$ be a continuous translation invariant operator on $\mathcal{S}(\mathbb{R})$. Then there is a $C^\infty$ function $F$ on $\mathbb{R}$ with all derivatives of polynomial growth at $\infty$ such that*

$(\star) \qquad \widehat{C\phi}(\xi) = F(\xi)\widehat{\phi}(\xi)$ *for all* $\phi \in \mathcal{S}(\mathbb{R})$.

*Conversely, every such function $F$ on $\mathbb{R}$ defines a continuous translation invariant operator $C$ on $\mathcal{S}(\mathbb{R})$ by $(\star)$.*

$G = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ locally compact group under multiplication

- Invariant integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,\frac{\mathrm{d}x}{|x|}$
- Characters $e_{\tau,\epsilon}(x) = \mathrm{sgn}(x)^{\epsilon}|x|^{\mathrm{i}\tau}$ with $\tau \in \mathbb{R}$ and $\epsilon = \pm 1$
  $\widehat{G} \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z})$
- Fourier-Mellin transform $\displaystyle\widehat{f}(\tau,\epsilon) = \int_{-\infty}^{\infty} f(x)e_{\tau,\epsilon}(x)\,\frac{\mathrm{d}x}{|x|}$
  for $f \in L^1(\mathbb{R}, \frac{\mathrm{d}x}{|x|})$
- Plancherel Formula
  $$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\frac{\mathrm{d}x}{|x|} = \sum_{\epsilon=\pm 1}\int_{-\infty}^{\infty} \widehat{f}(\tau,\epsilon)\overline{\widehat{g}(\tau,\epsilon)}\,\mathrm{d}\tau$$
  for $f, g \in L^2(\mathbb{R}, \frac{\mathrm{d}x}{|x|})$

Log Trick: Use group homomorphism $x \mapsto (\log|x|, \mathrm{sgn}(x))$ to turn Fourier-Mellin transform into Fourier transform on $\mathbb{R} \times (\mathbb{Z}/2\mathbb{Z})$.

$\mathbb{Q}_p =$ completion of $\mathbb{Q}$ relative to $p$-adic absolute value ($p$ prime)
$|p^k r/s|_p = p^{-k}$ if $r$, $s$ relatively prime to $p$

- $Q_p$ locally compact totally disconnected field with metric
  $d(x,y) = |x-y|_p, \qquad |x+y|_p = \max\{|x|_p, |y|_p\}$
- $p$-adic expansion $\quad x = \sum_{n=k}^{\infty} a_n p^n \quad a_n \in \{0,1,\ldots,p-1\}$
  $|x|_p = p^{-k}$ with $k = \min\{n \,:\, a_n \neq 0\}$ if $x \neq 0$
- Ring of $p$-adic integers $\quad \mathbb{Z}_p = \{|x|_p \leq 1\}$ (compact)

Characters of $G =$ additive group of $\mathbb{Q}_p$

- $e(x) = \exp(2\pi i z)$ with $z = \sum_{n<0} a_n p^n \in \mathbb{Q} \quad (x \in z + \mathbb{Z}_p)$
- $G \cong \widehat{G} = \{e_y\}_{y \in \mathbb{Q}_p}$ where $e_y(x) = e(xy)$ for $x, y \in \mathbb{Q}_p$
  (note that $xy \in \mathbb{Q} \mod \mathbb{Z}_p$ and $e(\mathbb{Z}_p) = 1$)
- Fourier transform on $G$ analogous to Fourier transform on $\mathbb{R}$

Fourier-Mellin transform on $\mathbb{Q}_p^\times$ more complicated than $\mathbb{R}^\times = \mathbb{Q}_\infty^\times$:

$\mathbb{Q}_p^\times \cong \{p^k\}_{k\in\mathbb{Z}} \times (\mathbb{Z}/(p-1)\mathbb{Z}) \times A$ with $A = \exp\{x \,:\, |x|_p < 1\}$

$\widehat{\{p^k\}_{k\in\mathbb{Z}}} \cong \mathbb{R}/\mathbb{Z}$ (compact) $\quad \widehat{A} \cong \varprojlim_{k\to\infty} \mathbb{Z}/(p^k\mathbb{Z})$ (countable)

$G =$ Euclidean motion group on $\mathbb{R}^n$ (translations and rotations)

$\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ Laplace operator on $\mathbb{R}^n$

Polynomials in $\Delta$ give all differential operators on $\mathbb{R}^n$ invariant under $G$

Problem: Diagonalize $\Delta$

Fourier Transform Method:

Use spherical coordinates on $\mathbb{R}^n$ (singularity at 0) and expansion in spherical harmonics. On radial functions get Fourier-Bessel transform (integral transform with Bessel function kernel).

Radon Transform Method:

Use integral transform that turns $\Delta$ into $(\partial/\partial p)^2$ on even functions of $p \in \mathbb{R}$ with parameter $\omega \in \mathbb{S}^{n-1}$ (no singularity). Then diagonalize by one-dimensional Fourier transform.

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \ : \ x \cdot x = 1\}$ unit sphere

$x \cdot y = x_1 y_1 + \cdots + x_n y_n$ inner product on $\mathbb{R}^n$

Hyperplane with oriented normal $\omega \in \mathbb{S}^{n-1}$ and height $p \in \mathbb{R}$:

$H(\omega, p) = \{x \in \mathbb{R}^n \ : \ x \cdot \omega = p\}$

Write $\xi = H(\omega, p) \cong \mathbb{R}^{n-1}$

$\mathrm{d}m = (n-1)$-dimensional Lebesgue measure on $\xi$

$\mathbb{P}^n =$ set of all hyperplanes $\xi$ in $\mathbb{R}^n$ (smooth $n$-dim manifold)

two-sheeted covering $\mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{P}^n$ (no singularities)

$(\omega, p) \mapsto H(\omega, p) = H(-\omega, -p)$

Radon transform of $f \in \mathcal{S}(\mathbb{R}^n)$:

$F(\omega, p) = \int_{x \cdot \omega = p} f(x)\, \mathrm{d}m(x)$

- Integral converges since $f|_{H(\omega,p)}$ is rapidly decreasing
- $F(\xi) = F(\omega, p)$ defined on $\mathbb{P}^n$ since $F(\omega, \xi) = F(-\omega, -\xi)$
- Fourier transform $\widehat{f}(r\omega) = \int_{-\infty}^{\infty} F(\omega, p) \mathrm{e}^{-2\pi \mathrm{i} rp}\, \mathrm{d}p$
- Radon transform of $\Delta f(x)$ is $(\partial/\partial p)^2\, F(\omega, p)$

For $x \in \mathbb{R}^n$
$K(x) =$ all hyperplanes $\xi$ containing $x$
$= \{(\omega, p) \ : \ x \cdot \omega = p\} \cong \mathbb{S}^{n-1}/\pm 1$
Let $\mathrm{d}\mu =$ invariant measure on $K(x)$ (total mass 1)
For $F \in \mathcal{S}(\mathbb{P}^n)$ define dual Radon transform

$$\widetilde{F}(x) = \int_{\xi \in K(x)} F(\xi)\,\mathrm{d}\mu(\xi) = \int_{\omega \in \mathbb{S}^{n-1}} F(\omega, \omega \cdot x)\,\mathrm{d}\omega$$

Radon Inversion Formula
If $f \in \mathcal{S}(\mathbb{R}^n)$ and $F =$ Radon transform of $f$, then

$$f(x) = c\,(-\Delta)^{(n-1)/2}\,\widetilde{F}(x) \qquad (c = \text{normalizing constant})$$

odd dimensions: Inversion formula is local - differential operator applied to $\widetilde{F}(x)$
even dimensions: Inversion formula is non-local - square root of differential operator (Hilbert transform) applied to $\widetilde{F}(x)$