James Taylor’s Oral Exam Syllabus

I. Functional Analysis
   A. Basics of Banach spaces
      i. Examples such as \( L^p \) spaces, sequences spaces, direct sums, and quotients
      ii. Linear functionals: duals, reflexive spaces, and the Hahn Banach theorem
      iii. Baire category, open maps, closed graphs, and Banach-Steinhaus
      iv. Basics of Hilbert spaces (polarization, adjoints, Riesz lemma)
   B. Useful topological notions
      i. Nets
      ii. Compactness (Tychonoff’s Theorem, Urysohn’s Lemma, Stone-Weierstrass Theorem)
      iii. Measure theory on compact spaces (Riesz-Markov Theorem)
      iv. Various topologies on operator spaces
      v. The Banach-Alaoglu Theorem
   C. Bounded Operator Theory
      i. Adjoints
      ii. Spectrum
      iii. Positive operators, square roots
      iv. Compact operators
      v. Fredholm operators and the Fredholm Alternative
      vi. Spectral Mapping Theorem
      vii. Functional Calculus
      viii. Various Spectral Theorems and measures related to self-adjoint operators
   D. Unbounded Operator Theory
      i. Definitions and generalizations
      ii. Self-adjoint, closed, essentially self-adjoint, and symmetric operators
      iii. Cayley Transform
      iv. Spectral Theorem for unbounded self-adjoint operators
      v. Stone’s Theorem
   E. Differential Calculus on Banach Spaces
      i. Derivative of operators on Banach spaces and generalizations of elementary differential calculus
      ii. Inverse and Implicit Function Theorems
      iii. Infinite Dimensional Manifolds and their issues
   F. Differential Operators and Spectral Theory
      i. Schwarz space
      ii. Fourier Transform
      iii. Distributions
      iv. Sobolev Spaces
      v. Various Laplace operators and their uses
   G. Applications to Quantum Mechanics
      i. Basics of Bohmian Mechanics and the formalism of Quantum Mechanics
      ii. Position, momentum operators
      iii. One-dimensional problems such as the harmonic oscillator, different potential wells
      iv. Spin
      v. Bell’s Theorem
II. Differential Geometry

A. Basic Definitions and examples
   i. Definitions of Manifolds, tangent vectors, vector fields, vector bundles
   ii. Examples: Surfaces, Lie groups-Matrix groups, submanifolds
   iii. Various mappings such as immersions, induced maps
   iv. Quotient manifolds: Projective spaces, Grassmann manifolds

B. Tensors and differential forms
   i. Tensors of all types, tensor fields, maps and tensors
   ii. Exterior algebra, exterior derivative, differential forms
   iii. Orientability and n-forms
   iv. Symmetrizing, alternating, contracting, and multiplying tensors
   v. Tensor Derivations
   vi. Lie Derivatives
   vii. Poincare Lemma and its partial converse

C. Vector fields
   i. Existence and Uniqueness Theorems for ODE
   ii. One-Parameter groups
   iii. Vector Fields as flows and as differential operators
   iv. Lie algebra of vector fields
   v. Frobenius' Theorem and foliations

D. Metrics and Connections
   i. Definition of Metrics and Connections
   ii. Covariant Derivative
   iii. The Levi-Civita connection
   iv. Parallel Translation
   v. Geodesics
   vi. Frame fields
   vii. Exponential Map
   viii. Hopf-Rinow Theorem
   ix. Curvature

E. Integration on Manifolds
   i. Definition of the integral
   ii. Manifolds with boundary
   iii. Stokes' Theorem

F. Surface theory
   i. Fundamental Forms, Gauss Curvature, Principal Curvature
   ii. The Gauss Theorem
   iii. Gauss-Bonnet Theorem