Problem Set 1, Solutions

(1) Find the space of polynomials $f(x) = ax^2 + bx + c$ satisfying $f(-1) = -1, f'(-1) = -1$. *Ans: The constraints give the linear system* $a - b + c = -1$, $-2a+b = -1$. *The system can be solved by Gaussian elimination: Adding twice the first equation to the second gives* $-b+2c = -3$ *so* $b = 2c+3$ *and* $a = (b+1)/2 = c+2$. *So the space of solutions is* $\{f(x) = (2 + c)x^2 + (2c + 3)x + c, c \in \mathbb{R}\}$.

(2) Show that the set V of functions $f : S \to \mathbb{R}$ from a set S to the real numbers $\mathbb{R}$ with addition given by $(f+g)(x) = f(x)+g(x)$ and scalar multiplication given by $(cf)(x) = c(f(x))$ satisfies axiom (VS4) in the axioms for a vector space. *Ans: We claim that for any* $f \in V$, *there exists a* $g \in V$ *such that* $f+g = 0$, *or in other words,* $f(x) + g(x) = 0$ *for all* $x \in S$. *Let* g *be the function given by* $g(x) = -f(x), x \in S$. *Then* $(f + g)(x) = f(x) + g(x) = f(x) - f(x) = 0$ *for all* $x \in S$ *as desired.*

(3) Show that the set of polynomials in a single real variable with addition given by $(f + g)(x) = f(x) + g(x)$ and scalar multiplication $(cf)(x) = f(cx)$ is not a vector space. *Ans. To show that* $(V, +, \cdot)$ *as above is not a vector space, it suffices to show that one axiom fails. We claim that the axiom VS8 fails, that is, there* $\exists f \in V, a, b \in \mathbb{R}$ *such that* $(a + b)f \neq af + bf$. *Indeed, take* $f(x) = x^2, a = 2$ *and* $b = 3$. *Then* $((a + b)f)(x) := f((a + b)x) = f(5x) = 25x^2$ *but* $(af + bf)(x) = (af)(x) + (bf)x = f(2x) + f(3x) = (2x)^2 + (3x)^2 = 13x^2$. *If* $x = 1$, *then* $25x^2$ *is not equal to* $13x^2$, *which shows that the two functions are not equal as claimed.*

(4) Show that if V is a vector space and $v \in V$ and $c \in F$ are such that $cv = 0$ then either $c = 0$ or $v = 0$. [Hint: argue by cases or by contradiction.] *Ans. Suppose that* $cv = 0$. *We claim that* $c = 0$ *or* $v = 0$. *If* $c = 0$, *then we are done. Otherwise if* $c \neq 0$ *then multiplying both sides by* $1/c$ *and using axiom VS6 and VS5 gives* $(1/c)(cv) = ((1/c)c)v = 1v = v$. *On the other hand, we showed in class that* $(1/c)0 = 0$. *Hence* $v = 0$ *which proves the claim in this case as well.*

(5) Prove that $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is linearly independent over $\mathbb{R}$ but linearly dependent over $\mathbb{Z}_2$ *Ans. Working with the field* $\mathbb{R}$, *the determinant of the resulting* 3×3 *matrix is* 2, *which shows linear independence in that case. On the other hand, if the field is* $\mathbb{Z}_2$ *then* $(1, 1, 0) + (1, 0, 1) + (0, 1, 1) = (2, 2, 2) = (0, 0, 0)$ *mod* 2 *which is a dependence relation, so the vectors are dependent in this case.*