Some review problems for the second exam:

(1) (a) Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $(Tv)_k = (v_{k+1} + v_{k-1})/2$ (where the indices are taken mod 3, so $v_4 = v_1, v_0 = v_3$.) (b) Find the inverse of T. (c) Find the determinant of T using row reduction. (d) Find the eigenvalues of T. (e) Find the diagonalization of T. (f) Let $v = [1\ 0\ 0]$. Use (e) to find a formula for $T^n v$ for any n.

(2)True or false: In each case prove your answer either way. (a) Let $T : V \to W$ be a linear transformation and $B \subset V$ a basis. If T is one-to-one then $T(B)$ is linearly independent. (b) If A is a square matrix then A and A^T have the same eigenvectors. (c) Any square matrix is a product of elementary matrices. (d) The eigenvalues of a symmetric matrix are all real. (e) Any skew-symmetric 2×2 matrix is diagonalizable. (f) $\det(A + B) = \det(A) + \det(B)$ for any square matrices of the same size. (g) $\det(P_1 P_2) = \det(P_1)\det(P_2)$ for any permutation matrices P_1, P_2. (Do not use the property $\det(AB) = \det(A)\det(B)$; the problem is to prove a special case of this property.) (h) Any two-dimensional inner product space has an orthonormal basis.

(3) True or false: In each case prove your answer either way. (a) Consider the linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n, e_1 \mapsto e_2, e_2 \mapsto e_3, \ldots, e_n \mapsto e_1$. Then T has n distinct eigenvalues. (b) The map $(f, g) \mapsto \int_0^1 x f(x)\overline{g(x)}dx$ defines an inner product on the space of continuous functions from 0 to 1. (c) For vectors v_1, v_2 in an inner product space, if $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$ then v_1 and v_2 are orthogonal. (d) Let V be the space of bounded functions from the integers $\mathbb{Z}$ to the complex numbers $\mathbb{C}$. The shift operator $(Tf)_k = f_{k-1}$ has infinitely many eigenvalues. (e) In an inner product space V, for any vector $x \in V$ we have $\langle 0, x\rangle = 0$. (f) In an inner product space V, for any subspace W, if $w \in W$ then the orthogonal projection of w onto W is equal to w.

(4) (a) Find an orthogonal basis for the subspace $W = \{x+y+z+w = 0\}$ in $\mathbb{R}^4$. (b) Find the linear function $a + bx$ that best approximates the quadratic function x^2 on the interval $[0, 2]$ with respect to the standard inner product on $C^\infty([0, 2])$. (That is, find the function $a + bx$ closest to x^2). Draw a graph of the function x^2 and the approximation you found. (c) (For additional practice) Make the set of functions $\{1, x^2, x^4\}$ on $[0, 1]$ into an orthogonal set with respect to the standard inner product on $C^\infty([0, 1])$.

(5) (a) Suppose that the reduced row echelon form of a matrix is $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. If the second and fourth columns of the matrix are $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ respectively, what is the matrix? (b) Find a basis for the null-space of the matrix? (c) Find a basis for the column-space of the matrix.

(6) (For additional practice) (a) Suppose that V is a vector space and U, W are subspaces so that $U + W = V$ and $U \cap W = \{0\}$. Show that any element of V can be written uniquely as a sum of elements of U and W. (b) Suppose that $S : U \to V$ and $T : V \to W$ are linear transformations. Show that $N(S) \subseteq N(T \circ S)$. (c) Let

A be a square matrix. Show that if all the columns sum to zero, then A is not invertible.