Some review problems for the second exam:

(1) (a) Find the matrix for the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $(Tv)_k = (v_{k+1} + v_{k-1})/2$ (where the indices are taken mod 3, so $v_4 = v_1, v_0 = v_3$.)
(b) Find the inverse of $T$. (c) Find the determinant of $T$ using row reduction. (d) Find the eigenvalues of $T$. (e) Find the diagonalization of $T$. (f) Let $v = [1 \ 0 \ 0]$. Use (e) to find a formula for $T^n v$ for any $n$.

(2) True or false: In each case prove your answer either way. (a) Let $T : V \rightarrow W$ be a linear transformation and $B \subset V$ a basis. If $T$ is one-to-one then $T(B)$ is linearly independent. (b) If $A$ is a square matrix then $A$ and $A^T$ have the same eigenvectors. (c) Any square matrix is a product of elementary matrices. (d) The eigenvalues of a symmetric matrix are all real. (e) In an inner product space $V$ is a subspace of linearly independent. (f) $\text{det}(A + B) = \text{det}(A) + \text{det}(B)$ for any square matrices of the same size. (g) $\text{det}(P_1 P_2) = \text{det}(P_1) \text{det}(P_2)$ for any permutation matrices $P_1, P_2$. (Do not use the property $\text{det}(AB) = \text{det}(A)\text{det}(B)$; the problem is to prove a special case of this property.) (h) Any two-dimensional inner product space has an orthonormal basis.

(3) True or false: In each case prove your answer either way. (a) Consider the linear transformation $T : \mathbb{C}^n \rightarrow \mathbb{C}^n, e_1 \mapsto e_2, e_2 \mapsto e_3, \ldots, e_n \mapsto e_1$. Then $T$ has $n$ distinct eigenvalues. (b) The map $(f, g) \mapsto \int_0^1 xf(x)g(x)dx$ defines an inner product on the space of continuous functions from 0 to 1. (c) For vectors $v_1, v_2$ in an inner product space, if $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$ then $v_1$ and $v_2$ are orthogonal. (d) Let $V$ be the space of bounded functions from the integers $\mathbb{Z}$ to the complex numbers $\mathbb{C}$. The shift operator $(Tf)_k = f_{k-1}$ has infinitely many eigenvalues. (e) In an inner product space $V$, for any vector $x \in V$ we have $\langle x, x \rangle = 0$. (f) In an inner product space $V$, for any subspace $W$, if $w \in W$ then the orthogonal projection of $w$ onto $W$ is equal to $w$.

(4) (a) Find an orthogonal basis for the subspace $W = \{x+y+z+w = 0\}$ in $\mathbb{R}^4$. (b) Find the linear function $a + bx$ that best approximates the quadratic function $x^2$ on the interval $[0, 2]$ with respect to the standard inner product on $C^\infty([0, 2])$. (That is, find the function $a + bx$ closest to $x^2$). Draw a graph of the function $x^2$ and the approximation you found. (c) (For additional practice) Make the set of functions $\{1, x^2, x^4\}$ on $[0, 1]$ into an orthogonal set with respect to the standard inner product on $C^\infty([0, 1])$.

(5) (a) Suppose that the reduced row echelon form of a matrix is \[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
If the second and fourth columns of the matrix are \[
\begin{bmatrix}
1 \\
2 \\
1 \\
\end{bmatrix}
\]
and \[
\begin{bmatrix}
1 \\
0 \\
1 \\
\end{bmatrix}
\]
respectively, what is the matrix? (b) Find a basis for the null-space of the matrix? (c) Find a basis for the column-space of the matrix.

(6) (For additional practice) (a) Suppose that $V$ is a vector space and $U, W$ are subspaces so that $U + W = V$ and $U \cap W = \{0\}$. Show that any element of $V$ can be written uniquely as a sum of elements of $U$ and $W$. (b) Suppose that $S : U \rightarrow V$ and $T : V \rightarrow W$ are linear transformations. Show that $N(S) \subseteq N(T \circ S)$. (c) Let
$A$ be a square matrix. Show that if all the columns sum to zero, then $A$ is not invertible.