Homework Sets # 7 and #8 , Math 311:02, Fall 2008
2,6,9,13,14,31,42,Sample Solutions

§3.1 #2 Task Define function f on $[-4,0]$ by

$$f(x)=\begin{cases}\frac{2x^2-18}{x+3} & \text{if} \quad x\neq -3\\ -12 & \text{if} \quad x=-3\end{cases}$$

Show that f is continuous at -3.
Proof Note that -3 is an accumulation point of $Dom(f)$ and $-3\in Dom(f)$. To prove that f is continuous at -3 in this situation it is sufficient to prove that $\lim_{x\to -3} f(x)=f(-3)$.

When $x\in Dom(f)-\{-3\}$ we have

$$f(x)=\frac{2x^2-18}{x+3}=\frac{2(x-3)(x+3)}{x+3}=2(x-3)\frac{x+3}{x+3}=2x-6$$

since $x+3\neq 0$. The polynomial $2x-6$ is continuous everywhere, and in particular at -3. So

$$\lim_{x\to -3} f(x)=\lim_{x\to -3}(2x-6)=2(-3)-6=-12$$

Since $f(-3)=-12$, we have shown

$$\lim_{x\to -3} f(x)=-12=f(-3)$$

and it follows that f is continuous at -3.

§3.1 #6 Task Let $f(x)=\sqrt{x}$. Show that f is continuous at each p in $Dom(f)$.

Proof We know that $Dom(f)$ is the set of non-negative reals. Suppose that $p\geq 0$. Consider an arbitrary positive ε. We need to provide a positive δ with the property that

$$\text{for all non-negative } x,\ |x-p|<\delta \Rightarrow \left|\sqrt{x}-\sqrt{p}\right|<\varepsilon. \qquad (6.1)$$

We treat the case where $p=0$ separately from the case $p>0$.

Case 1: $p=0$

Since $p=0$, we know that, for all non-negative x,

$$\left|\sqrt{x}-\sqrt{p}\right|<\varepsilon \Leftrightarrow \sqrt{x}<\varepsilon \Leftrightarrow 0\leq\sqrt{x}<\varepsilon \Leftrightarrow 0\leq x<\varepsilon^2.$$

So we take $\delta=\varepsilon^2$. Clearly δ is positive and condition (6.1) is satisfied.

Case 2: $p>0$

For non-negative x we have

$$\sqrt{x}-\sqrt{p}=\frac{\sqrt{x}-\sqrt{p}}{1}\cdot\frac{\sqrt{x}+\sqrt{p}}{\sqrt{x}+\sqrt{p}}=\frac{x-p}{\sqrt{x}+\sqrt{p}}.$$

From this we get a global Lipschitz condition at p

$$\left|\sqrt{x}-\sqrt{p}\right|=\left|\frac{x-p}{\sqrt{x}+\sqrt{p}}\right|\leq\frac{|x-p|}{\sqrt{p}}=\frac{1}{\sqrt{p}}\cdot|x-p|.$$

Note that

$$\frac{1}{\sqrt{p}}\cdot|x-p|<\varepsilon \Leftrightarrow |x-p|<\sqrt{p}\cdot\varepsilon.$$

Take $\delta = \sqrt{p} \cdot \varepsilon$. Now consider arbitrary x in $Dom(f)$. Assume that $|x-p| < \delta$. Then

$$\left|\sqrt{x} - \sqrt{p}\right| \le \frac{1}{\sqrt{p}} \cdot |x-p| < \frac{1}{\sqrt{p}}.\delta = \varepsilon.$$

§3.1 #9 TASK Define f on $(0,1)$ by $f(x) = x\sin(1/x)$. Can one define $f(0)$ so that f is continuous at 0? Explain.
Result Yes, if one defines $f(0) = 0$ then the resulting function f with Dom(f) now equal to the interval $[0,1)$ will be continuous at 0.
Proof For x in the enlarged domain $[0,1)$ we get the estimate

$$x \neq 0 \Rightarrow |f(x) - f(0)| = |x\sin(1/x) - 0| = |x| \cdot |\sin(1/x)| \le |x| = |x-0|$$

and thus we get a Lipschitz condition at 0 :

$$|f(x) - f(0)| \le 1 \cdot |x-0|$$

Continuity at 0 follows.

§3.2 #13 Task Suppose that f has domain D; $p \in D$; and f is continuous at p. Show that There is a positive M and a neighborhood Q of p such that $|f(x)| \le M$ for all x in $Q \cap D$.

Work Note that I prefer p to the book's x_o so I don't have to type subscripts. Note also that the simplest neighborhoods we know are the open intervals of the form $V_r(p) = (p-r, p+r)$. So that will be what kind of neighborhood I look for.

Try $\varepsilon = 1$. By the continuity assumption, we get a positive δ so that

$$\text{for all } x \text{ in } D, \quad |x-p| < \delta \Longrightarrow |f(x) - f(p)| < \varepsilon.$$

Take the neighborhood $Q = (p-\delta, p+\delta)$. Take $M = |f(p)| + 1$. Consider an arbitrary x in $Q \cap D$. Assume further that $|x-p| < \delta$. Now we use the triangle inequality to get

$$|f(x)| = |f(x) - f(p) + f(p)| \le |f(x) - f(p)| + |f(p)| < 1 + |f(p)| = M$$

So our Q and M have the desired property.■

§3.2 #14 TASK Suppose that f is continuous at p in $Dom(f)$. Set $g(x) = |f(x)|$. Show that g is continuous at p.

Proof. Consider an arbitrary postive ε. Since f is continuous at p, we can and do pick a postive δ such that, for all x in $Dom(f)$

$$|x-p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon. \tag{14.1}$$

Use this δ to work with g. Note that $Dom(g) = Dom(f)$.
Consider arbitrary x in $Dom(g)$. So we know $x \in Dom(f)$.
Assume that $|x-p| < \delta$.
Recall that for all real a and b

$$|\,|a| - |b|\,| \le |a-b|.$$

Thus

$$|g(x) - g(p)| = |\,|f(x)| - |f(p)|\,| \le |f(x) - f(p)| < \varepsilon$$

by using (14.1).

§3.2 #31 This problem uses the definition of closed. Some of the students may have omitted it.
Task. Suppose that functions f and g are both continuous on their common domain $[a,b]$. Let $T = \{x \text{ in } [a,b] : f(x) = g(x)\}$. Show that T is closed.

Work. By definition T is closed iff every accumulation point for T is in T. So we start by considering an arbitrary accumulation point p for T. Thus we know that every open neighborhood of p contains infinitely many points of T and thus at least one point of T other than p itself. Every set of the form $(p - 1/n\ ,\ p + 1/n)$ is a neighborhood of p. For each positive integer n pick x_n from the infinite set $T \cap (p - 1/n\ ,\ p + 1/n) - \{p\}$. Now we see that

$f(x_n)$ converges to $f(p)$ since $\lim (x_n) = p$; all $x_n \in Dom\,(f)$; and f is continuous at p

$g(x_n)$ converges to $g(p)$ since $\lim (x_n) = p$; all $x_n \in Dom\,(g)$; and g is continuous at p

$f(p) = \lim f(x_n) = \lim g(x_n) = g(p)$ since all $x_n \in T$, and f and g agree on each point of T

Thus $p \in T$.

§3.2 #42 Find an interval of length 1 that contains a root of $x^3 - 6x^2 + 2.826 = 0$.

Result One such interval is $[0, 1]$.

Work Here we use the IVT. We look for an interval of length 1 on which the polynomial function defined $P(x) = x^3 - 6x^2 + 2.826$ changes sign. Note that

$$\begin{aligned} P(0) &= 2.826 > 0 \\ P(1) &= 1 - 6 + 2.826 < 1 - 6 + 3 = -2 < 0 \end{aligned}$$

So P changes sign on $[0, 1]$.