Sample Solutions for Midterm Exam #2, Math 311:03, Fall 2009

Note: The solutions to Part 1 appear on the last page.

Task 2.1 [20 points]
a. [10 points] Consider the function f defined by the formula $f(x) = 1/x$. Give an ε-δ-proof that

$$\lim_{x \to 3} f(x) = \frac{1}{3}$$

Proof: Consider an arbitrary positive ε. Note that $Dom(f) = \mathbb{R} - \{3\}$. For arbitrary x in the domain of f ,

$$|f(x) - f(3)| = \left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| = \frac{|x-3|}{3\,|x|}.$$

If x can be near 0, then this quantity can be very big. So the first thing to do in finding our δ is to set $\delta_1$equal to some positive quantity which is small enough that $|x-3| < \delta_1$ implies that x cannot be near zero. For example, take $\delta_1 = 2$ and keep $|x-3| < \delta_1$. Now we have

$$0 < 1 = 3 - \delta_1 < x < 3 + \delta_1 = 4.$$

Since x is now positive

$$|x| = x > 1 \qquad \text{and} \qquad \frac{1}{|x|} = \frac{1}{x} < \frac{1}{1} = 1.$$

To summarize, whenever $|x-3| < \delta_1$ we have

$$|f(x) - f(3)| = \frac{|x-3|}{3\,|x|} < \frac{1}{3} \cdot |x-3|\,. \tag{1}$$

Note that for all x,

$$\frac{1}{3} \cdot |x-3| < \varepsilon \iff |x-3| < 3\varepsilon.$$

Now we take $\delta = \min(1, 3\varepsilon)$ and note that for all x in $Dom(f)$, whenever $|x-3| < \delta$ then

$$\begin{aligned} |f(x) - f(3)| &< \frac{1}{3} \cdot |x-3| \qquad \text{using (1) since } |x-3| < \delta \le \delta_1 \\ &< \frac{1}{3} \cdot 3\varepsilon = \varepsilon \qquad \text{since } |x-3| < \delta \le 3\varepsilon \end{aligned}$$

b. [10 points] Function g defined by the formula $g(x) = \sin(1/x)$.
0 is an accumulation point of $Dom(g)$. (Do not prove this fact.)
Explain why g does not have a limit at 0.

Reasoning By the sequential criterion for sequential limits it suffices to find a sequence of imputs $(x_x)_{n\in\mathbb{N}}$ in $Dom(g)$ such that $\lim(x_n) = 0$ and $\lim(g(x_n))$ does not exist. The graph of g shows infinitely many oscillations between $+1$ and -1 as x decreases toward zero. This suggests considering the sequence with $x_n = 2/n\pi$. Clearly, $\lim(x_n) = 0$. We get

$$g(x_n) = \sin(n\pi/2) = \begin{cases} \sin(0) = 0 & \text{if } n \equiv 0 \mod 2 \\ \sin(\pi/2) = 1 & \text{if } n \equiv 1 \mod 4 \\ \sin(3\pi/2) = -1 & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Thus the sequence of outputs $(g(x_n))_{n\in\mathbb{N}}$ has no limit since it has subsequences with different limits. Note that

$$(g(x_n))_{\mathbb{N}} = (1, 0, -1, 0, 1, 0, -1, 0, 1, ...)$$

Task 2.2 [30 points]

Consider the function g defined by

$$g(x) = \begin{cases} 8x & \text{if } x \in \mathbb{Q} \\ 2x^2 + 8 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

a. [10] Prove that g is not continuous at 0.
Proof: Note that $0 \in \mathbb{Q} \subseteq \mathbb{R} = Dom(g)$.

By the sequential criterion for continuity at a point we can show that g is not continuous at 0 by displaying a sequence of inputs $(x_n)_{\mathbb{N}}$ such that $\lim(x_n) = 0$ but $\lim(g(x_n)) \neq g(0)$.

Take $x_n = \pi/n$. All these x_n are irrational. They clearly converge to zero. We have

$$\lim g(x_n) = \lim\left[2(x_n)^2 + 8\right] = \lim\left[2\left(\frac{\pi}{n}\right)^2 + 8\right] = 2\cdot 0^2 + 8 = 8 \neq 0 = g(0).$$

b. **[10]** Find one positive real M such that for all x in $[1,3]$,

$$|g(x)-16| \le M \ \ |x-2|\,.$$

Show your work clearly. (No proof is required - just clear work.)
Result: $M =$ __10__
Work: Suppose that $x \in [1,3]$. Then $1 \le x \le 3$. Either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$.
If $x \in \mathbb{Q}$ then we have $|g(x) - g(2)| = |8x - 2 \cdot 8| = 8\,|x-2|$
If $x \notin \mathbb{Q}$ then we have

$$\begin{aligned} |g(x) - g(2)| &= \left|(2x^2+8) - (2\cdot 2^2 + 8)\right| = 2\left|x^2-4\right| \\ &= 2\,|x+2| \ \ |x-2| \end{aligned}$$

Since we have $1 \le x \le 3$ we see that $0 < 3 \le x+2 \le 5$ and thus that $|x+2| = x+2 < 5$. So in the case $x \notin \mathbb{Q}$ we get $|g(x) - g(2)| \le 2\cdot 5\cdot |x-2|$.

Regardless of the rationality of x, we see that $|g(x)-g(2)| \le \max(8,10)\cdot |x-2|$.

c. **[10]** Show that g is continuous at 2.

Consider an arbitrary positive ε. Take $\delta = \min(1, \varepsilon/10)$.
I will verify that for all x in $Dom(g)$, $|x-2| < \delta \Longrightarrow |g(x) - g(2)| < \varepsilon$.
Consider an arbitrary x in $Dom(g)$.
Assume that $|x-2| < \delta$.
Since $|x-2| < \delta \le 1$ we know that $x \in [1,3]$ and the estimate from part b is valid.

$$|g(x) - g(2)| \le 10 \ \ |x-2|$$

But $|x-2| < \delta \le \varepsilon/10$ so we have $10 \ \ |x-2| < 10 \cdot \varepsilon/10 = \varepsilon$. Using the transitivity property we get $|g(x)-g(2)| < \varepsilon$

Task 2.3 [18]
a. [12] Suppose that p is a real number and that $1 < p < 32$.
Explain why the equation $x^5 - p = 0$ must have a solution in $(1, 2)$.

Reasoning. Consider the polynomial function $f(x) = x^5 - p$. Since all polynomials are continuous, f is continuous on $\mathbb{R}$. Apply the Intermediate Value Theorem to f on the closed bounded interval $[1, 2]$. Note that $f(1) = 1 - p < 0$ since $1 < p$. Note that $f(2) = 32 - p > 0$ since $p < 32$. Thus there must be a real number c such that $1 < c < 2$ and $f(c) = 0$. This c is a solution of the equation $x^5 - p = 0$ in $(1, 2)$.

b. [6] Now suppose that q is a real number with $1 < q$.
Explain why there must be a solution of $x^5 - q = 0$ in $(1, q)$.

Reasoning

Here we consider a (possibly different) polynomial function $f(x) = x^5 - q$.
This polynomial is continous on all of $\mathbb{R}$.
So we can apply the intermediate value theorem to f on $[1, q]$.
Note that $f(1) = 1 - q < 0$ by assumption.
Note that $f(q) = q^5 - q = q\left(q^4 - 1\right) > 0$ since $q > 1 > 0$ and $q^4 > 1$.
Thus there must be a real c such that $1 < c < q$ and $f(c) = 0$.
NOTE This demonstrates the existence of a positive fifth root of q.

Task 2.4 [9] Suppose that

a function f is defined and continuous on $[-2,0]$

a function g is defined and continuous on $[0,2]$.

and $f(0)=g(0)$.

Define a function h on $[-2,2]$ by

$$h(x)=\begin{cases} f(x) & \text{if } x\leq 0 \\ g(x) & \text{if } x\geq 0 \end{cases}$$

Give an ε-δ-proof that h is continuous at 0.

Proof Consider an arbitrary positve ε.

Note that $Dom(h)=[-2,2]$.

It suffices to display a postive δ with the property that for all x in $Dom(h)$

$$|x-0|<\delta \Longrightarrow |h(x)-h(0)|<\varepsilon.$$

Since f is continuous at 0 we get a positive δ_1 so that

$$\text{for all } x \text{ in } [-2,0],\ \ |x-0|<\delta_1 \Longrightarrow |f(x)-f(0)|<\varepsilon.$$

Since g is continuous at 0 we get a positive δ_2 so that

$$\text{for all } x \text{ in } [0,2],\ \ |x-0|<\delta_2 \Longrightarrow |g(x)-g(0)|<\varepsilon.$$

Take $\delta=\min(\delta_1,\delta_2)$. Verify that δ has the required property. Suppose that x is an arbitrary element of $[-2,2]$. Assume that $|x-0|<\delta$.

Case 1: $x\leq 0$ Then we know that $x\in Dom(f)=[-2,0]$ and that $|x-0|<\delta\leq\delta_1$. Recall that $h(0)=f(0)$. So we get

$$|h(x)-h(0)|=|f(x)-f(0)|<\varepsilon \quad \text{by the choice of } \delta_1$$

Case 2: $x>0$ Then we know that $x\in Dom(g)=[0,2]$ and that $|x-0|<\delta\leq\delta 2$. Recall that $h(0)=g(0)$. So we get

$$|h(x)-h(0)|=|g(x)-g(0)|<\varepsilon \quad \text{by the choice of } \delta_2$$

So in any case, $|h(x)-h(0)|<\varepsilon$.

Definitions

Recall that a real number p is an *accumulation point* for set S iff there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $S - \{p\}$ which converges to p.

a. Function f is has a *limit at point* p iff there is a number L such that
(1) p is an accumulation point of $Dom(f)$ and
(2) for every positive ε there is a positive δ such that for all x in $Dom(f)$

$$0 < |x - p| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

b. Function f is *continuous at point* p iff (1) $p \in Dom(f)$ and
(2) for every positive ε there is a positive δ such that for all x in $Dom(f)$

$$|x - p| < \delta \Longrightarrow |f(x) - f(p)| < \varepsilon.$$

c. Function f is *continuous on set* S iff f is continuous at each point p in S.

d. Function f is *continuous* iff f is continuous at each point p in $Dom(f)$.

Theorems.

a. *The sequential criterion for functional limits:*
Suppose that f is a function, that $p \in \mathbb{R}$, and that $L \in \mathbb{R}$. $\lim_{x\to p} f(x) = L$ iff (1) p is an accumulation point of $Dom(f)$, and (2) for every sequence $(x_n)_{n\in\mathbb{N}}$ in $Dom(f) - \{p\}$, $\lim(x_n) = p \Longrightarrow \lim\left(f\left(x_n\right)\right) = L$.

b. *The sequential criterion for continuity at a point.* Suppose that f is a function and $p \in \mathbb{R}$. f is continuous at p iff
(1) $p \in Dom(f)$ and (2) for every sequence $(x_n)_{n\in\mathbb{N}}$ in $Dom(f)$, $\lim(x_n) = p \Longrightarrow \lim\left(f\left(x_n\right)\right) = f(p)$.

c. *The Intermediate Value Theorem*
Suppose that f is continuous on an interval $[a, b]$ with $a < b$.
If the real number y is strictly between $f(a)$ and $f(b)$
then there is a real number c such that $a < c < b$ and $f(c) = y$.