Midterm Exam #1, Math 311:03, Fall 2009

Sample Solutions

The solutions to Part 1 appear on the last page of this packet.

Task 2.1 [16 points]

Suppose that $S \subseteq \mathbb{R}$, $A = lub(S)$, and $T = \{-a : a \in \mathbf{S}\}$

a. [8 points] Show that $-A$ is a lower bound for T.

Proof: We must show that every t in T satisfies the condition $-A \leq t$.

Consider an arbitrary t in T. This t can be expresses as $t = -a$ for some a in S . We know that $a \leq lub(S) = A$ and that $-a \geq -A$. Thus $-A \leq -a = t$.

b. [8] Show that if C is an arbitrary lower bound for T, then $C \leq -A$.

Proof: It is sufficient to show that $-C \geq A$.

Consider an arbitrary a in S. We know both that $-a \in T$ and $a \leq A$. Since C is a lower bound for T, $-a \geq C$ and thus $a \leq -C$. Thus $-C$ is an upper bound for S and $A = lub(S) \leq -C$.

Task 2.2 [16 points] Suppose that $A = lub(S)$ but $A \notin S$.

a. Show that there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbf{S}$ such that

$$\text{for all indices, } |a_n - A| < \frac{1}{n}$$

Proof: We must show how to select each a_n to form the sequence.

Consider an arbitrary positive integer n. Note that $A - 1/n < A$. Thus $A - 1/n$ cannot be an upper bound for the set and therefore there must be an element of S which is larger than $A - 1/n$. Pick one such element and call it a_n.

Now for each n we have

$$a_n \leq A < A + \frac{1}{n} \text{ since } A \text{ is an upper bound for } S \text{ and } \frac{1}{n} > 0$$

$$A - \frac{1}{n} < a_n \text{ since that was how we chose } a_n.$$

So

$$A - \frac{1}{n} < a_n < A + \frac{1}{n} \quad \text{and thus} \quad |a_n - A| < \frac{1}{n}.$$

Task 2.3 [22 points]

a. [11] Using theorems about arithmetic properties of sequences explain why

$$\lim\left(4\cdot\frac{3n+6}{9n+3}\right)=\frac{4}{3}$$

Explanation

Consider an arbitrary n in $\mathbb{N}$. Let $q_n=\frac{3n+6}{9n+3}$. Divide top and bottom by n. Note that now

$$q_n=\frac{t_n}{b_n}\quad\text{where } t_n=3+6\cdot\frac{1}{n}\text{ and } b_n=9+3\cdot\frac{1}{n}$$

We know that $\lim(1/n)=glb(1/n)=0$. By results on limits of constant sequences and limits of sums and limits of products we get

$$\lim(t_n)=3+6\cdot 0=3\quad\text{and}\quad\lim(b_n)=9+3\cdot 0=9.$$

Now use the theorem on limits of quotients noting that $\lim(b_n)\neq 0$ to get

$$\lim(q_n)=\frac{3}{9}=\frac{1}{3}$$

and use the theorem on limits of constant multiples to get

$$\lim\left(4\cdot\frac{3n+6}{9n+3}\right)=4\cdot\lim(q_n)=4\cdot\frac{1}{3}=\frac{4}{3}$$

b. [11] Give an $\varepsilon-N$ proof that

$$\lim\frac{2n-3}{5n-4}=\frac{2}{5}$$

Proof: We must show that for each positive ε there is a positive integer N such that whenever $n\geq N$ then also

$$\left|\frac{2n-3}{5n-4}-\frac{2}{5}\right|<\varepsilon.$$

We do some preliminary "simplification". For all indices

$$\frac{2n-3}{5n-4}-\frac{2}{5}=\frac{5(2n-3)-(5n-4)2}{5(5n-4)}=\frac{(10n-15)-(10n-8)}{25n-20}=\frac{-7}{25n-20}.$$

Note that the bottom of the last fraction on the right is always positive since $n \geq 1$. Thus, still for all indices we have

$$\left|\frac{2n-3}{5n-4} - \frac{2}{5}\right| = \left|\frac{-7}{25n-20}\right| = \frac{7}{25n-20}.$$

For all indices we have

$$\frac{7}{25n-20} < \varepsilon \Leftrightarrow \frac{7}{\varepsilon} < 25n - 20 \Leftrightarrow \frac{(7/\varepsilon)+20}{25} < n.$$

We complete the proof by taking

$$N = \left\lfloor \frac{(7/\varepsilon)+20}{25} \right\rfloor + 1.$$

Whenever $n \geq N$ we also have

$$\frac{(7/\varepsilon)+20}{25} < n \quad \text{and thus} \quad \left|\frac{2n-3}{5n-4} - \frac{2}{5}\right| = \frac{7}{25n-20} < \varepsilon.$$

Task 2.4 [16] Suppose that

$(H1)$ $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences

$(H2)$ for all indices, $|a_n| \le 60$ and $|b_n| \le 80$

Using only the definition of Cauchy sequence (that is, without using any general theorems about Cauchy sequences) show that

(R) the product sequence $(a_n b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof: Consider an arbitrary postive ε. We need to provide a positive integer H so that whenever both $j \ge H$ and $k \ge H$ then also $|a_j b_j - a_k b_k| < \varepsilon$.

First we note that for all indices

$$(1) \quad \begin{aligned} |a_j b_j - a_k b_k| &= |(a_j - a_k)\, b_j - a_k\,(b_j - b_k)| \\ &\le |a_j - a_k|\ |b_j| + |a_k|\ |b_j - b_k| \le 80\,|a_j - a_k| + 60\,|b_j - b_k|\,. \end{aligned}$$

Set

$$\varepsilon_1 = \frac{\varepsilon}{160} \quad \text{and } \varepsilon_2 = \frac{\varepsilon}{120}.$$

By (H1) we get positive integers H_1 and H_2 such that

whenever both $j \ge H_1$ and $k \ge H_1$ then also (2) $|a_j - a_k| < \varepsilon_1$ and

whenever both $j \ge H_2$ and $k \ge H_2$ then also (3) $|b_j - b_k| < \varepsilon_2$.

Now choose $H = \max(H_1, H_2)$. Suppose both $j \ge H$ and $k \ge H$. We then have

$j \ge H_1$ and $k \ge H_1$ so inequality (2) holds and

$j \ge H_2$ and $k \ge H_2$ so inequality (3) holds.

From these and the inequality (1) we see that whenever both $\lfloor y \rfloor j \ge H$ and $k \ge H$ we get both (2) and (3) and thus

$$|a_j b_j - a_k b_k| < 80 \cdot \frac{\varepsilon}{160} + 60 \cdot \frac{\varepsilon}{120} = \varepsilon.$$

Task 2.5 [15 points]

For a real y, use the notation

$$\begin{aligned} \lfloor y \rfloor &= \text{the largest integer which is not bigger than } y \\ &= \max\{m \ : \ m \in \mathbb{Z} \text{ and } m \le y\}. \end{aligned}$$

Thus, for every real y, we have $\lfloor y \rfloor \le y < \lfloor y \rfloor + 1$.

Consider an arbitrary real x.

a. **[5]** Suppose that k is an arbitrary positive integer.
Explain why $k\lfloor x\rfloor \le \lfloor kx\rfloor$.

WORK Both k and $\lfloor x\rfloor$ are integers. So the product $k\lfloor x\rfloor$ is an integer. Since $\lfloor x\rfloor \le x$ and $k > 0$ we get $k\lfloor x\rfloor \le kx$. Thus

$$k\lfloor x\rfloor \in \{m \text{ in } \mathbb{Z} \ : \ m \le kx\}$$

and

$$k\lfloor x\rfloor \le \max\{m \text{ in } \mathbb{Z} \ : \ m \le kx\} = \lfloor kx\rfloor.$$

b. [5] Suppose that k is an arbitrary positive integer.
Explain why $kx - k < \lfloor kx\rfloor < kx + 1$.
You may use the result of (a), whether or not you finished (a).

WORK By the definition of the greatest integers function we get

$$\lfloor kx\rfloor \le kx < \lfloor kx\rfloor + 1$$

Using only the first of those inequalities we get $\lfloor kx\rfloor \le kx < kx + 1$ so $\lfloor kx\rfloor < kx + 1$.

By definition again we get $x < \lfloor x\rfloor + 1$. Multiply by k. We get $kx < k\lfloor x\rfloor + k$ and thus $kx - k < k\lfloor x\rfloor \le \lfloor kx\rfloor$. Thus $kx - k < \lfloor kx\rfloor < kx + 1$.

For each n in $\mathbb{N}$, set

$$a_n = \frac{\lfloor x\rfloor + \lfloor 2\ x\rfloor + ... + \lfloor n\ x\rfloor}{n^2} = \frac{\sum_{k=1}^{n}\lfloor k\ x\rfloor}{n^2}$$

c. **[5]** Show that $\lim(a_n) = x/2$.

NOTE As a result of a typographical error, the inequality in (b) is not quite strong enough to get the desired result. HINT The sum of the first n positive integers is $n(n+1)/2$.

We work first with the inequality $\lfloor kx\rfloor < kx + 1$, which is valid for all k in $\mathbb{N}$ ..

$$\begin{aligned} a_n &= \frac{\sum_{k=1}^{n}\lfloor k\ x\rfloor}{n^2} < \frac{\sum_{k=1}^{n}(k\ x + 1)}{n^2} = \frac{x\sum_{k=1}^{n} k\ + \sum_{k=1}^{n} 1}{n^2} \\ &= \frac{x}{n^2}\frac{n(n+1)}{2} + \frac{n}{n^2} = \frac{x}{2}\frac{n+1}{n} + \frac{1}{n} \end{aligned}$$

Using the order properties of limits we get $\lim(a_n) \le x/2$, assuming that the limit exits..

Using the inequality $kx - k < \lfloor kx \rfloor$ we can get only

$$\frac{x}{2}\frac{n\,(n+1)}{n^2} - \frac{n\,(n+1)}{2n^2} < a_n$$

which gets us only

$$\frac{x}{2} - \frac{1}{2} \leq \lim(a_n) \ \text{ assuming that the limit exists.}$$

However we know that $\lfloor kx \rfloor \leq kx < \lfloor kx \rfloor + 1$ from which we get a better inequality

$$(b') \qquad kx - 1 < \lfloor kx \rfloor$$

From this we get

$$a_n = \frac{\sum_{k=1}^{n} \lfloor k\ x \rfloor}{n^2} > \frac{\sum_{k=1}^{n} (kx - 1)}{n^2} = \frac{x}{2}\frac{n\,(n+1)}{n^2} - \frac{n}{n^2} = \frac{x}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{n}$$

Now we can apply the squeeze theorem to the inequalities

$$\frac{x}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{n} < a_n < \frac{x}{2}\left(1 + \frac{1}{n}\right) + \frac{1}{n}$$

to get the existence and value of the limit.

#1, Solutions to Part 1, Math 311:03, Fall 2009

Task 1.1 Complete the following definitions

a. Supose that $S \subseteq \mathbb{R}$ and $A \in \mathbb{R}$.
We say that "A is the least upper bound of S" if and only if
$A = \min(\ \mathcal{UB}(S)\ )$
Recall that $B \in \mathcal{UB}(S) \Longleftrightarrow$ for all s in S, $s \leq B$

b. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers.
We say that this sequence is *bounded* if and only if
there exists at least one positive M such that $\forall n$ in $\mathbb{N}$, $|x_n| \leq M$.

c. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers and that $L \in \mathbb{R}$.
We say that "$\lim(x_n) = L$" if and only if

$$\forall\varepsilon \textit{ in } \mathbb{R}^+,\ \exists N \text{ in } \mathbb{N} \text{ such that } \forall n \text{ in } \mathbb{N},\ n \geq N \Longrightarrow |x_n - L| < \varepsilon.$$

d. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers.
We say that "$(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence*" if and only if

$$\forall\varepsilon \textit{ in } \mathbb{R}^+,\ \exists H \text{ in } \mathbb{N} \text{ such that } \forall j,k \text{ in } \mathbb{N},\ [j \geq H \text{ and } k \geq H] \Longrightarrow |x_j - x_k| < \varepsilon$$

Task 1.2.

a. *The Axiom of Completeness*
Suppose that S is a non-empty subset of $\mathbb{R}$
If S is bounded above,
then there exists a real number A such that $A = lub(S)$.

b. *The Monotone Convergence Theorem* (my preferred version)
Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$.

If $(a_n)_{n\in\mathbb{N}}$ is decreasing and bounded below
then $(a_n)_{n\in\mathbb{N}}$ converges to its greatest lower bound.

If $(a_n)_{n\in\mathbb{N}}$ is increasing and bounded above
then $(a_n)_{n\in\mathbb{N}}$ converges to its least upper bound.

c. The *Bolzano-Weierstrass* Theorem (my preferred version)

Every bounded sequence of real numbers contains a convergent subsequence.