

1 Math 311:03 Fall 2009 The Exponential Series

1.1 Introduction

An "infinite series" is a formal sum of infinitely many terms. A series can be expressed as

$$\begin{aligned}\sum_{k=0}^{\infty} t_k &= t_0 + t_1 + t_2 + \dots + t_k + \dots \\ \sum_{k=1}^{\infty} t_k &= t_1 + t_2 + t_3 + \dots + t_k + \dots \\ \sum_{k=K}^{\infty} t_k &= t_K + t_{K+1} + t_{K+2} + \dots + t_k + \dots\end{aligned}$$

The "terms" in the series are the numbers being added — just as the factors in a product are the numbers being multiplied. So each series has a sequence in individual terms. In the first series above, the sequence of terms is $(t_k)_{k=0}^{\infty}$. In the other two series, the sequence of terms is indexed on different index sets. We can always relabel the terms to index on the non-negative integers. Note that

$$\sum_{k=K}^{\infty} t_k = t_{K+0} + t_{K+1} + t_{K+2} + \dots + t_k + \dots = \sum_{\ell=0}^{\infty} t_{K+\ell}$$

You can think of this process as making a change of index setting $\ell = k - K$.

Next we need to make analytic sense of the formal series — to say when the formal sum actually represents a well-defined real number.

Definitions Consider the formal series $\sum_{k=0}^{\infty} t_k$.

The " n^{th} partial sum" for this series is the finite sum

$$S_n = \sum_{k=0}^n t_k$$

Recall that the summation notation has the inductive definition

$$S_1 = \sum_{k=0}^1 t_k = t_0 + t_1 \quad \text{and} \quad S_{n+1} = S_n + t_{n+1}$$

We say that "the series $\sum_{k=0}^{\infty} t_k$ converges to a real number S " if and only if

the sequence of partial sums converges to S

or equivalently

$$\lim (S_n) = \lim \left(\sum_{k=0}^n t_k \right) = S.$$

When the series does converge to S we write

$$\sum_{k=0}^{\infty} t_k = S.$$

We say that an infinite series is "convergent" if and only if there is a real S to which the series converges. We say the series "diverges" if and only if the series does not converge to any real sum.

1.2 Problems (not to turn in, but to learn from)

These problems establish the basic arithmetic of convergent series.

1. Suppose that

$$\sum_{k=0}^{\infty} t_k = S \quad \text{and} \quad r \in \mathbb{R}.$$

Then (a) the sequence of individual terms must converge to zero, that is

$$\lim(t_k) = 0$$

and (b)

$$\sum_{k=0}^{\infty} r t_k = r S = r \left(\sum_{k=0}^{\infty} t_k \right).$$

2. Suppose that

$$\sum_{k=0}^{\infty} t_k = S \quad \text{and} \quad \sum_{k=0}^{\infty} r_k = R.$$

Then (a)

$$\text{for all indices } n, \quad \sum_{k=0}^n (t_k + r_k) = \left(\sum_{k=0}^n t_k \right) + \left(\sum_{k=0}^n r_k \right)$$

and (b)

$$\sum_{k=0}^{\infty} (t_k + r_k) = \left(\sum_{k=0}^{\infty} t_k \right) + \left(\sum_{k=0}^{\infty} r_k \right)$$

3. Consider the formal series

$$\sum_{k=0}^{\infty} t_k \quad \text{and} \quad \sum_{k=0}^{\infty} r_k.$$

(a) Show that if $t_k \leq r_k$ for all the indices k , then for all positive integers n

$$\sum_{k=0}^n t_k \leq \sum_{k=0}^n r_k$$

(b) Show that if $0 \leq t_k \leq r_k$ for all indices AND the formal series of larger terms converges then so does the sum of the smaller terms and

$$0 \leq \sum_{k=0}^{\infty} t_k \leq \sum_{k=0}^{\infty} r_k.$$

(c) Show that if $0 \leq t_k \leq r_k$ for all indices AND the sum of the smaller terms diverges so does the sum of the larger terms.

(d) Give examples to show that the part of the hypothesis " $0 \leq t_k$ for all k " is necessary in (b) and (c).

4. Consider the formal series

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

Show that for all n in \mathbb{N}

$$\sum_{k=0}^n \frac{1}{k!} \leq 1 + \sum_{\ell=0}^n \left(\frac{1}{2} \right)^{\ell}$$

We concentrate on the following series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Note that there is a real variable x in each of the terms. Thus for each different real x we have a different series to consider.

We emphasize the following two conventions for this topic

x^0 denotes the number 1 regardless of the value of x

$0!$ denotes the integer 1.

Since the terms in the series depend on x , so do the partial sums. We write

$$S_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

1.3 Convergence

Theorem For each real x there is a real number, which we will denote by $E(x)$, such that our series converges to $E(x)$.

Proof: We proceed by cases.

Case 1: $x = 0$ In this case we see that

$$\begin{aligned} S_1(0) &= \sum_{k=0}^1 \frac{0^k}{k!} = 1 + 0 = 1 \\ S_2(0) &= \sum_{k=0}^2 \frac{0^k}{k!} = 1 + 0 + 0 = 1 \\ S_{n+1}(0) &= \sum_{k=0}^{n+1} \frac{0^k}{k!} = \sum_{k=0}^n \frac{0^k}{k!} + \frac{0^{n+1}}{(n+1)!} = S_n(0) + 0 = 1 \end{aligned}$$

Thus for all positive n

$$S_n(0) = 1.$$

Thus the sequence of partial sums is the sequence with all terms equal to 1 and

$$\sum_{k=0}^{\infty} \frac{0^k}{k!} = \lim (S_n(0)) = \lim (1, 1, 1, \dots) = 1$$

So we have

$$E(0) = \sum_{k=0}^{\infty} \frac{0^k}{k!} = 1.$$

Case 2: $x > 0$ Consider a fixed positive x .

Since x is positive, so are all integer powers of x . Thus we see

$$S_{n+1}(x) = S_n(x) + \frac{x^{n+1}}{(n+1)!} > S_n(x).$$

Our sequence of partial sums is increasing. By the Monotone Convergence Theorem, to show that it converges, it will be enough to find an upper bound for the sequence of partial sums. Since x is fixed, our upper bound will probably depend in some way on that fixed x .

Before doing the general case I will practice with two particular positive values for x . The first has particular importance for calculus and probability.

Subcase: $x = 1$ By the Homework problem 4 above we know that, for each positive n ,

$$S_n(1) = 1 + \sum_{k=1}^n \frac{1^k}{k!} \leq 1 + \sum_{k=1}^n \left(\frac{1}{2}\right)^{k-1} = 1 + \sum_{\ell=0}^{n-1} \left(\frac{1}{2}\right)^\ell$$

By earlier work with finite geometric series we know, for $n \geq 1$, that

$$\sum_{\ell=0}^{n-1} \left(\frac{1}{2}\right)^\ell = \frac{1 - (1/2)^n}{1 - (1/2)} < \frac{1}{1 - (1/2)} = 2.$$

Thus we have

$$S_n(1) < 1 + 2 = 3 \text{ for all the } n \text{ in } \mathbb{N}.$$

So we have our upper bound for the increasing sequence of partial sums. Thus we get the existence of

$$E(1) = \sum_{k=0}^{\infty} \frac{1^k}{k!}$$

and further we get the estimates

$$2.5 = S_2(1) \leq E(1) = \text{lub}(\{S_n(1)\}) \leq 3.$$

Subcase $x = 7$ This case has no interest in its own right, but it illustrates the method we use in the general case.

We know that the sequence of partial sums $(S_n(7))_1^\infty$ is increasing. So we need only find an upper bound. Suppose that n is much bigger than 7.

$$S_n(7) = \sum_{k=0}^n \frac{7^k}{k!} = \underbrace{\sum_{k=0}^7 \frac{7^k}{k!}}_{\text{Call this } A} + \sum_{k=8}^n \frac{7^k}{k!}$$

Note that A is independent of n . Now for $k \geq 8$

$$\frac{7^k}{k!} = \frac{7^7 \cdot 7^{k-7}}{1 \cdot 2 \cdot \dots \cdot 7 \cdot 8 \cdot 9 \cdot \dots \cdot k} = \frac{7^7}{7!} \cdot \frac{7^{k-7}}{8 \cdot 9 \cdot \dots \cdot k}$$

Note that the expression $8 \cdot 9 \cdot \dots \cdot k$ has exactly $7 - k$ factors, each of which is at least 8. Thus

$$8 \cdot 9 \cdot \dots \cdot k \geq 8^{k-7} \quad \text{and} \quad \frac{1}{8 \cdot 9 \cdot \dots \cdot k} \leq \frac{1}{8^{k-7}}.$$

It follows that

$$\begin{aligned} \frac{7^k}{k!} &= \frac{7^7}{7!} \cdot \frac{7^{k-7}}{8 \cdot 9 \cdot \dots \cdot k} \leq \frac{7^7}{7!} \cdot \left(\frac{7}{8}\right)^{k-7} \\ \sum_{k=8}^n \frac{7^k}{k!} &\leq \sum_{k=8}^n \frac{7^7}{7!} \cdot \left(\frac{7}{8}\right)^{k-7} = \frac{7^7}{7!} \cdot \sum_{k=8}^n \left(\frac{7}{8}\right)^{k-7} \end{aligned}$$

Note that

$$\begin{aligned} \frac{7^7}{7!} \cdot \sum_{k=8}^n \left(\frac{7}{8}\right)^{k-7} &= \frac{7^7}{7!} \cdot \left[\left(\frac{7}{8}\right)^1 + \left(\frac{7}{8}\right)^2 + \dots + \left(\frac{7}{8}\right)^{n-7} \right] \\ &\leq \frac{7^7}{7!} \cdot \left[\left(\frac{7}{8}\right)^0 + \left(\frac{7}{8}\right)^1 + \left(\frac{7}{8}\right)^2 + \dots + \left(\frac{7}{8}\right)^{n-7} \right] \\ &\leq \frac{7^7}{7!} \cdot \left[\frac{1}{1 - (7/8)} \right] \end{aligned}$$

So we see that for large n , say $n \geq 10$, we have the bound

$$S_n(7) \leq A + \frac{7^7}{7!} \cdot \left[\frac{1}{1 - (7/8)} \right]$$

Of course for smaller n we have

$$S_n(7) \leq S_{10}(7) \leq A + \frac{7^7}{7!} \cdot \left[\frac{1}{1 - (7/8)} \right]$$

Thus our increasing sequence of partial sums is bounded from above and must converge. The limit is our

$$E(7) = \sum_{k=0}^{\infty} \frac{7^k}{k!}.$$

Treating arbitrary positive x . First we pick a integer M such that $x \leq M$. For all n we get

$$S_n(x) = \sum_{k=0}^n \frac{x^k}{k!} \leq \sum_{k=0}^n \frac{M^k}{k!} = S_n(M).$$

Here we used one of those homework problems. An upper bound for the partial sums $S_n(M)$ will thus be an upper bound for the partial sums $S_n(x)$.

We know that the sequence of partial sums $(S_n(M))_1^{\infty}$ is increasing. So we need only find an upper bound. Suppose that n is bigger than M .

$$S_n(M) = \sum_{k=0}^n \frac{M^k}{k!} = \underbrace{\sum_{k=0}^M \frac{M^k}{k!}}_{\text{Call this } A} + \sum_{k=M+1}^n \frac{M^k}{k!}$$

Note that A is independent of n . Now for $k > M$

$$\frac{M^k}{k!} = \frac{M^M \cdot M^{k-M}}{1 \cdot 2 \cdot \dots \cdot M \cdot (M+1) \cdot (M+2) \cdot \dots \cdot k} = \frac{M^M}{M!} \cdot \frac{M^{k-7}}{(M+1) \cdot (M+2) \cdot \dots \cdot k}$$

Note that the expression $(M+1) \cdot (M+2) \cdot \dots \cdot k$ has exactly $M-k$ factors, each of which is bigger than M . Thus

$$\begin{aligned} (M+1) \cdot (M+2) \cdot \dots \cdot k &\geq (M+1)^{M-k} \quad \text{and} \\ \frac{1}{(M+1) \cdot (M+2) \cdot \dots \cdot k} &\leq \frac{1}{(M+1)^{M-k}}. \end{aligned}$$

It follows that

$$\frac{M^k}{k!} = \frac{M^M}{M!} \cdot \frac{M^{k-M}}{(M+1) \cdot (M+2) \cdot \dots \cdot k} \leq \frac{M^M}{M!} \cdot \left(\frac{M}{M+1} \right)^{k-M}$$

and

$$\sum_{k=M+1}^n \frac{M^k}{k!} \leq \sum_{k=8}^n \frac{M^M}{M!} \cdot \left(\frac{M}{M+1} \right)^{k-M} = \frac{M^M}{M!} \cdot \sum_{k=M+1}^n \left(\frac{M}{M+1} \right)^{k-M}.$$

Set $r = M/(M+1)$. Note that $0 < r < 1$ and

$$\begin{aligned} \sum_{k=M+1}^n \left(\frac{M}{M+1} \right)^{k-M} &= \cdot \left[(r)^1 + (r)^2 + \dots + (r)^{n-M} \right] \\ &\leq \left[(r)^0 + (r)^1 + (r)^2 + \dots + (r)^{n-M} \right] \\ &\leq \left[\frac{1}{1 - M/(M+1)} \right] = \frac{M+1}{M} \end{aligned}$$

So we see that for large n , say $n \geq M + 1$, we have the bound

$$S_n(M) \leq A + \frac{M^M}{M!} \cdot \left[\frac{M+1}{M} \right]$$

Of course for $n \leq M$ we have

$$S_n(M) \leq S_{M+1}(7) \leq A + \frac{M^M}{M!} \cdot \left[\frac{M+1}{M} \right]$$

Thus our increasing sequence of partial sums $S_n(M)$ is bounded from above and must converge.

$$E(M) = \sum_{k=0}^{\infty} \frac{M^k}{k!}.$$

But since now $E(M)$ is an upper bound for all the $S_n(x)$ we also have

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim (S_n(x)) \leq \lim (S_n(M)) = E(M) = \sum_{k=0}^{\infty} \frac{M^k}{k!}.$$

Case 3 $x < 0$ The main difficulty in this case is that the partial sums are not monotone. One way to proceed is to use the Nested Interval Theorem. Another is to make a no-frills argument using only Monotone Convergence Theorem and an auxilliary series. A third method uses the theorem that all Cauchy sequences are convergent.

Applying the Nested Interval Theorem.

I will illustrate the Nested Interval Theorem method in the case $-1 < x < 0$. It is more complicated to make the argument for general negative x . We will rely on the other two methods for the general case.

Assume that x is a fixed arbitrary real such that $-1 < x < 0$.

Claim 1: The subsequence of partial sums with even subscripts is decreasing, i.e.,

$$\text{for all positive integers } m, S_{2m}(x) > S_{2m+2}(x)$$

Proof: Consider an arbitrary m in \mathbb{N} . It is sufficient to show that $S_{2m+2}(x) - S_{2m}(x) < 0$. Now

$$\begin{aligned} S_{2m+2}(x) - S_{2m}(x) &= \sum_{k=2m+1}^{2m+2} \frac{x^k}{k!} = \frac{x^{2m+1}}{(2m+1)!} + \frac{x^{2m+2}}{(2m+2)!} \\ &= \frac{x^{2m+1}}{(2m+1)!} \left[1 + \frac{x}{2m+2} \right] = \frac{-|x|^{2m+1}}{(2m+1)!} \left[1 + \frac{x}{2m+2} \right] \end{aligned}$$

The first factor is negative. The second factor, the one in square brackets is positive because $-1 < x < 0$ and thus

$$\left[1 + \frac{x}{2m+2} \right] = 1 - \frac{|x|}{2m+2} > 1 - \frac{1}{2m+2} > 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

Thus the difference $S_{2m+2}(x) - S_{2m}(x)$ is indeed negative.

Claim 2: The subsequence of partial sums with odd subscripts is increasing, i.e.,

$$\text{for all positive integers } m, S_{2m+1}(x) > S_{2m-1}(x)$$

Proof: Consider an arbitrary m in \mathbb{N} . It is sufficient to show that $S_{2m+1}(x) - S_{2m-1}(x) > 0$. Now

$$\begin{aligned}
S_{2m+1}(x) - S_{2m-1}(x) &= \sum_{k=2m}^{2m+1} \frac{x^k}{k!} = \frac{x^{2m}}{(2m)!} + \frac{x^{2m+1}}{(2m+1)!} \\
&= \frac{x^{2m}}{(2m)!} \left[1 + \frac{x}{2m+1} \right] = \frac{|x|^{2m}}{(2m)!} \left[1 + \frac{x}{2m+1} \right]
\end{aligned}$$

The first factor is positive. The second factor, the one in square brackets is positive because $-1 < x < 0$ and thus

$$\left[1 + \frac{x}{2m+1} \right] = 1 - \frac{|x|}{2m} > 1 - \frac{1}{2m} \geq 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

Thus the difference $S_{2m+1}(x) - S_{2m-1}(x)$ is indeed positive.

Claim 3 For every m in \mathbb{N} , $S_{2m-1} < S_{2m}$.

Proof: Consider arbitrary positive m .

$$S_{2m} - S_{2m-1} = \frac{x^{2m}}{(2m)!} > 0$$

since $x < 0$ and thus $x^{2m} > 0$.

Claim 4 For all m and ℓ in \mathbb{N} , $S_{2m-1}(x) < S_{2\ell}(x)$.

Proof: We work by cases.

If $m = \ell$, then we simply appeal to Claim 3.

If $m < \ell$, then we appeal to Claims 2 and 3:

$$S_{2m-1}(x) < S_{2\ell-1}(x) < S_{2\ell}(x).$$

If $\ell < m$, then we appeal to Claims 1 and 3

$$S_{2m-1}(x) < S_{2m}(x) < S_{2\ell}(x).$$

Claim 5 $\lim(S_n(x))$ exists.

Proof: We use the method of proof of the Nested Interval Theorem. Since our x is fixed, we write S_n to abbreviate $S_n(x)$.

Since we know that the partial sums for odd subscripts are increasing and bounded above by S_2

$$\lim(S_{2m-1}) \text{ exists and equals } \text{lub}\{S_{2m-1} : m \in \mathbb{N}\}.$$

Since we know that the partial sums for even subscripts are decreasing and are bounded below by S_1

$$\lim(S_{2m}) \text{ exists and equals } \text{glb}\{S_{2m} : m \in \mathbb{N}\}.$$

Since $S_{2m} = S_{2m-1} + (-1)^{2m}/(2m)!$ and $\lim_{2m \rightarrow \infty} (-1)^{2m}/(2m)! = 0$

$$\lim(S_{2m-1}) = \lim(S_{2m}).$$

Let's write

$$L = \lim(S_{2m-1}) = \lim(S_{2m}).$$

Thus for all m

$$S_{2m-1} \leq L \leq S_{2m} = S_{2m-1} + \frac{x^{2m}}{(2m)!}$$

We now can finish by showing that

$$L = \lim(S_n)$$

Consider an arbitrary positive ε . We will display a positive integer N such that

$$\text{whenever } n \geq N \text{ then also } |S_n - L| < \varepsilon.$$

From the convergence of the subsequences discussed above, we get N_1 and N_2 such that

$$\begin{aligned} \text{whenever } m &\geq N_1 \text{ then also } |S_{2m-1} - L| < \varepsilon \\ \text{whenever } m &\geq N_2 \text{ then also } |S_{2m} - L| < \varepsilon \end{aligned}$$

Set $N = 2 \times \max(N_1, N_2)$. Suppose that $n \geq N$.

If n is even, we write $n = 2m$ and note that $m \geq \max(N_1, N_2) \geq N_2$ so

$$|S_n - L| = |S_{2m} - L| < \varepsilon.$$

However if n is odd, we write $n = 2m - 1$ and note that $2m - 1 \geq 2 \max(N_1, N_2)$ and thus $2m \geq 2 \max(N_1, N_2) + 1$ and thus $m \geq N_1$ and

$$|S_n - L| = |S_{2m-1} - L| < \varepsilon.$$

So in either case we get $|S_n - L| < \varepsilon$.

Applying "brute force".

We fix an arbitrary negative x and set $p = |x|$. Recall that $x = -p$.

Let's start with two practice computations. First look at $n = 5$.

$$\begin{aligned} S_5(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \\ &= 1 - p + \frac{p^2}{2!} - \frac{p^3}{3!} + \frac{p^4}{4!} - \frac{p^5}{5!} \quad \text{and} \\ S_5(p) &= 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \frac{p^4}{4!} + \frac{p^5}{5!} \end{aligned}$$

Adding we get

$$S_5(x) + S_5(p) = 2 + 2\frac{p^2}{2!} + 2\frac{p^4}{4!} = 2 \left(1 + \frac{p^2}{2!} + \frac{p^4}{4!} \right)$$

Next look at $n = 6$.

$$\begin{aligned} S_6(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \\ &= 1 - p + \frac{p^2}{2!} - \frac{p^3}{3!} + \frac{p^4}{4!} - \frac{p^5}{5!} + \frac{p^6}{6!} \quad \text{and} \\ S_6(p) &= 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \frac{p^4}{4!} + \frac{p^5}{5!} + \frac{p^6}{6!}. \end{aligned}$$

Adding we get

$$S_6(x) + S_6(p) = 2 + 2\frac{p^2}{2!} + 2\frac{p^4}{4!} + \frac{p^6}{6!} = 2 \left(1 + \frac{p^2}{2!} + \frac{p^4}{4!} + \frac{p^6}{6!} \right).$$

We can verify that, for all positive integers n ,

$$S_n(x) + S_n(p) = 2 A_n \stackrel{\text{notation}}{=} \begin{cases} 2 \sum_{k=0}^{n/2} \frac{p^{2k}}{(2k)!} & \text{if } n \text{ is even} \\ 2 \sum_{k=0}^{(n-1)/2} \frac{p^{2k}}{(2k)!} & \text{if } n \text{ is odd} \end{cases}$$

Another way to understand the terms A_n is this

$$\begin{aligned} \text{for } n \text{ even, } A_n &= \sum_{\ell \text{ is even and } \ell \leq n} \frac{p^\ell}{\ell!} \\ \text{for } n \text{ odd, } A_n &= \sum_{\ell \text{ is even and } \ell \leq n-1} \frac{p^\ell}{\ell!} \end{aligned}$$

So if n is odd, then $n + 1$ is even and

$$A_{n+1} = A_n$$

However if n is even, then $n + 1$ is odd and

$$A_{n+1} = A_n + \frac{p^{2n+2}}{(2n+2)!}.$$

The value of this is that the sequence $(A_n)_1^\infty$ is increasing. By adding the "missing" odd powers of p , we see that

$$A_n < \sum_{k=0}^n \frac{p^k}{k!} < E(p) \quad \text{for all positive integers } n.$$

This tells us that the sequence $(A_n)_1^\infty$ does converge. Since

$$S_n(x) = A_n - S_n(p)$$

and both sequences on the right are convergent we see that

$$E(x) = \lim (S_n(x)) \text{ exists.}$$

This finishes the existence proof.

Applying the Cauchy Criterion for convergent sequences.

Here we use our earlier theorem that a sequence in \mathbb{R} converges to a limit in \mathbb{R} if and only if the sequence is Cauchy.

We fix our negative x and again write $p = |x|$. Since $p > 0$ we know that the partial sums $S_n(p)$ for the series $E(p)$ are Cauchy.

Now we prove that the sequence of partial sums $S_n(x)$ for the series $E(x)$ is Cauchy.

Consider an arbitrary positive ε . Since the sequence $(S_n(p))_{n \in \mathbb{N}}$ is Cauchy we get a positive integer H such that

$$\text{whenever } m \geq H \text{ and } n \geq H \text{ then } |S_n(p) - S_m(p)| < \varepsilon.$$

Use this H . Suppose that $m \geq H$ and $n \geq H$. I want to verify that $|S_n(x) - S_m(x)| < \varepsilon$.

Case $m = n$ Then

$$|S_n(x) - S_m(x)| = 0 < \varepsilon$$

Case $m < n$ Here we have

$$|S_n(x) - S_m(x)| = \left| \sum_{k=m+1}^n \frac{x^k}{k!} \right| \leq \sum_{k=m+1}^n \frac{|x|^k}{k!} = S_n(p) - S_m(p)$$

and thus

$$|S_n(x) - S_m(x)| \leq S_n(p) - S_m(p) = |S_n(p) - S_m(p)| < \varepsilon.$$

Case $m > n$ Here we have

$$|S_n(x) - S_m(x)| = \left| \sum_{k=n+1}^m \frac{x^k}{k!} \right| \leq \sum_{k=n+1}^m \frac{|x|^k}{k!} = S_m(p) - S_n(p)$$

and thus

$$|S_n(x) - S_m(x)| \leq S_m(p) - S_n(p) = |S_m(p) - S_n(p)| < \varepsilon.$$

This completes the proof that the series $E(x)$ converges for each real x . ■

We now use the notation $E(x)$ for both the formal series and the real number which is its sum.

Remark 1

For any x and n

$$|S_n(x)| = \left| \sum_{k=0}^n \frac{x^k}{k!} \right| \leq \sum_{k=0}^n \frac{|x|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = E(|x|).$$

Remark 2

For any x

$$|E(x)| = |\lim(S_n(x))| = \lim |S_n(x)| \leq E(|x|)$$

Remark 3

If $0 < a < b$, then

$$1 < E(a) < E(b)$$

Proof of the third remark. To see the first inequality we note that

$$E(a) = \lim(S_n(a)) = \text{lub}(\{S_n(a) : n \in \mathbb{N}\}) \geq S_1(a) = 1 + a > 1$$

To get the second inequality we must be a bit more careful. For each big n

$$E(b) \geq S_n(b) = 1 + b + \sum_{k=2}^n \frac{b^k}{k!}$$

Homework Verify that for all n bigger than 1

$$\sum_{k=2}^n \frac{b^k}{k!} > \sum_{k=2}^n \frac{a^k}{k!}$$

Homework Verify the following for sequential limits.

$$\begin{aligned} \lim \sum_{k=0}^n \frac{b^k}{k!} &= 1 + b + \lim \sum_{k=2}^n \frac{b^k}{k!} \\ \lim \sum_{k=0}^n \frac{a^k}{k!} &= 1 + a + \lim \sum_{k=2}^n \frac{a^k}{k!} \\ \lim \sum_{k=0}^n \frac{b^k}{k!} &> \lim \sum_{k=0}^n \frac{a^k}{k!} \end{aligned}$$

being careful not to misstate the order properties for convergent sequences.

These "homework" problems were inadvertently proved in class.

This finishes the proof of the third remark.

1.4 Continuity

Theorem The function E is continuous at each point p in \mathbb{R}

Proof

First we attend to the case $p = 0$. Consider an arbitrary positive ε . As usual we seek a positive δ such that for all x in the domain, \mathbb{R} , of our function E ,

$$|x - 0| < \delta \implies |E(x) - E(0)| < \varepsilon.$$

Recall that we know $E(0) = 1$. Consider an arbitrary real x .

$$E(x) - E(0) = [\lim S_n(x)] - 1 = \lim [S_n(x) - 1]$$

For all n

$$\begin{aligned} S_n(x) - 1 &= \left(1 + \sum_{k=1}^n \frac{x^k}{k!}\right) - 1 = \sum_{k=1}^n \frac{x^k}{k!} \\ &= x \sum_{k=1}^n \frac{x^{k-1}}{k!} \end{aligned}$$

Now let's keep $|x - 0| < 1$. We'll use this constraint in the summation but not with the x we factored out.

$$|S_n(x) - 1| = |x| \sum_{k=1}^n \frac{|x|^{k-1}}{k!} \leq |x - 0| \cdot \sum_{k=1}^n \frac{1}{k!} \leq |x - 0| \cdot E(1).$$

Note that

$$|x - 0| \cdot E(1) < \varepsilon \iff |x - 0| < \frac{\varepsilon}{E(1)}.$$

Set $\delta = \min(\{1, \varepsilon/E(1)\})$. Clearly $\delta > 0$. Suppose that $x \in \mathbb{R} = \text{Dom}(E)$ and $|x - 0| < \delta$. Then since $|x - 0| < \delta \leq 1$

$$|S_n(x) - 1| \leq |x - 0| \cdot E(1)$$

and since $|x - 0| < \delta \leq \varepsilon/E(1)$

$$|x - 0| \cdot E(1) < \varepsilon.$$

So we see that for $x \in \text{Dom}(E)$

$$|x - 0| < \delta \implies |E(x) - E(0)| < \varepsilon.$$

Consider next an arbitrary real p . In this case p is not necessarily positive.

As usual we consider an arbitrary positive ε and look for a positive δ with the desired property.

Treat first an arbitrary real x

$$E(x) - E(p) = \lim S_n(x) - \lim S_n(p) = \lim (S_n(x) - S_n(p)).$$

For each n

$$\begin{aligned} S_n(x) - S_n(p) &= \sum_{k=0}^n \frac{x^k}{k!} - \sum_{k=0}^n \frac{p^k}{k!} = \left(1 + \sum_{k=1}^n \frac{x^k}{k!}\right) - \left(1 + \sum_{k=1}^n \frac{p^k}{k!}\right) \\ &= \sum_{k=1}^n \frac{x^k}{k!} - \sum_{k=1}^n \frac{p^k}{k!} = \sum_{k=1}^n \frac{x^k - p^k}{k!} \end{aligned}$$

and thus

$$(*) \quad |S_n(x) - S_n(p)| \leq \sum_{k=1}^n \frac{|x^k - p^k|}{k!} = |x - p| + \sum_{k=2}^n \frac{|x^k - p^k|}{k!}.$$

We now keep x in the closed bounded interval $[p - 1, p + 1]$. Note that

$$-|p| - 1 \leq p - 1 \leq x \leq p + 1 \leq |p| + 1.$$

Consider k with $k \geq 2$. We apply the Mean value theorem to the function taking x to x^k . This function has derivative kx^{k-1} . If x is not equal to p we get a c strictly between x and p such that

$$\frac{x^k - p^k}{x - p} = kc^{k-1} \quad \text{and} \quad x^k - p^k = k c^{k-1} (x - p).$$

Since c is between x and p we know that c is in $[-|p| - 1, |p| + 1]$ and thus $|c| \leq |p| + 1$. It follows that

$$|x^k - p^k| = k |c|^{k-1} |x - p| \leq k (|p| + 1)^{k-1} |x - p|$$

This last estimate is also valid when $x = p$ since then all it says is $0 \leq 0$.

Thus whenever $|x - p| \leq 1$, we can use this observation in the estimate $(*)$ to get

$$\begin{aligned} |S_n(x) - S_n(p)| &\leq |x - p| + \sum_{k=2}^n \frac{|x^k - p^k|}{k!} \\ &\leq |x - p| + \sum_{k=2}^n \frac{k (|p| + 1)^{k-1} |x - p|}{k!} \\ &\leq |x - p| \left(1 + \sum_{k=2}^n \frac{k (|p| + 1)^{k-1}}{k!} \right) \end{aligned}$$

Set $\ell = k - 1$ and note that

$$\begin{aligned} \left(1 + \sum_{k=2}^n \frac{k (|p| + 1)^{k-1}}{k!} \right) &= \left(1 + \sum_{k=2}^n \frac{(|p| + 1)^{k-1}}{(k-1)!} \right) \\ &= \left(1 + \sum_{\ell=1}^{n-1} \frac{(|p| + 1)^\ell}{\ell!} \right) = S_{n-1}(|p| + 1) \leq E(|p| + 1). \end{aligned}$$

Thus whenever $|x - p| \leq 1$

$$|S_n(x) - S_n(p)| \leq |x - p| E(|p| + 1).$$

Thus we can see that we can choose $\delta = \min(\{1, \varepsilon/E(|p| + 1)\})$. For all x in $Dom(E)$ if we assume $|x - p| < \delta$ then we get

$$|S_n(x) - S_n(p)| \leq |x - p| E(|p| + 1) < \delta \cdot E(|p| + 1) \leq \varepsilon.$$

1.5 Differentiability

Theorem At each point p in \mathbb{R} , the function E is differentiable and $E'(p) = E(p)$.

Proof Consider first an arbitrary positive integer n with $n \geq 3$. The function $S_n(x)$ is a polynomial in x . So we can compute its derivative

$$\begin{aligned} \frac{d}{dx} [S_n(x)] &= \frac{d}{dx} \left[\sum_{k=0}^n \frac{1}{k!} x^k \right] = \frac{d}{dx} \left[1 + x + \sum_{k=2}^n \frac{1}{k!} x^k \right] \\ &= 0 + 1 + \sum_{k=2}^n \frac{k}{k!} x^{k-1} = \sum_{k=1}^n \frac{1}{(k-1)!} x^{k-1} \\ &= \sum_{k=0}^{n-1} \frac{1}{k!} x^k = S_{n-1}(x) \end{aligned}$$

We also see that

$$\begin{aligned}\frac{d}{dx} [S_2(x)] &= \frac{d}{dx} \left[1 + x + \frac{x^2}{2} \right] = 0 + 1 + x = S_1(x) \\ \frac{d}{dx} [S_1(x)] &= \frac{d}{dx} [1 + x] = 0 + 1\end{aligned}$$

If we introduce the notation $S_0(x) = 1$ for all x , then we get the nice result that

$$\text{for all } n \text{ in } \mathbb{N}, \text{ the derivative of } S_n(x) \text{ is } S_{n-1}(x).$$

Consider a fixed n with $n \geq 2$. Keep x different from p . The mean value theorem tells us that

$$\frac{S_n(x) - S_n(p)}{x - p} - S_{n-1}(p) = S_{n-1}(c_n) - S_{n-1}(p)$$

for some c_n that lies between x and p . Applying the mean value theorem again

$$\frac{S_n(x) - S_n(p)}{x - p} - S_{n-1}(p) = S_{n-1}(c_n) - S_{n-1}(p) = S_{n-2}(\gamma_n) \cdot (c_n - \gamma_n)$$

where the point γ_n lies between c_n and p and thus also lies between x and p .

Now let's keep $0 < |x - p| < 1$. This gives us

$$-|p| - 1 \leq p - 1 < x < p + 1 \leq |p| + 1$$

Since

$$c_n \text{ lies between } x \text{ and } p$$

and

$$\gamma_n \text{ lies between } c_n \text{ and } p$$

we must have

$$\begin{aligned}-1 - |p| < p < \gamma_n < c_n < x < p + 1 \leq |p| + 1 & \text{ or} \\ -1 - |p| < p - 1 \leq x < c_n < \gamma_n < p < |p| + 1\end{aligned}$$

So in either case $|\gamma_n| \leq |p| + 1$. We also get

$$|c_n - \gamma_n| < |x - p|.$$

We now get the string of estimates

$$\begin{aligned}\left| \frac{S_n(x) - S_n(p)}{x - p} - S_{n-1}(p) \right| &= |S_{n-2}(\gamma_n) \cdot (c_n - \gamma_n)| \leq |S_{n-2}(\gamma_n)| \cdot |x - p| \\ &\leq E(|\gamma_n|) \cdot |x - p| \leq E(|p| + 1) \cdot |x - p|\end{aligned}$$

Now take the sequential limit as n runs to ∞ .

$$\begin{aligned}\lim \left(\left| \frac{S_n(x) - S_n(p)}{x - p} - S_{n-1}(p) \right| \right) &= \left| \lim \left(\frac{S_n(x) - S_n(p)}{x - p} - S_{n-1}(p) \right) \right| \\ &= \left| \frac{E(x) - E(p)}{x - p} - E(p) \right|\end{aligned}$$

and using the order properties we get

$$\left| \frac{E(x) - E(p)}{x - p} - E(p) \right| \leq E(|p| + 1) \cdot |x - p|$$

The right hand side goes to zero as $x \rightarrow p$. The squeeze theorem for functional limits now tells us that

$$\lim_{x \rightarrow p} \left| \frac{E(x) - E(p)}{x - p} - E'(p) \right| = 0$$

and this is equivalent to the statement that

$$\lim_{x \rightarrow p} \frac{E(x) - E(p)}{x - p} \text{ exists and equals } E'(p).$$

1.6 Consequences of Differentiability

Theorem For all x , $E(x) \cdot E(-x) = 1$.

Proof Use the product rule and the chain rule to compute

$$\begin{aligned} \frac{d}{dx} [E(x)E(-x)] &= E'(x)E(-x) + E(x)(E'(-x) \cdot -1) \\ &= E'(x)E(-x) - E(x)E'(-x) = 0 \end{aligned}$$

Since the function taking x to $E(x)E(-x)$ has constant derivative zero, it must be a constant function. So for all x

$$E(x)E(-x) = E(0)E(-0) = E(0)E(0) = 1 \cdot 1 = 1.$$

Theorem For all x

$$E(x) > 0$$

Proof We already know that $E(0) = 1$ and for all positive x , $E(x) > 1$.

Consider a negative x . So $x = -|x|$ and $-x = |x| > 0$. Now

$$E(x) = \frac{1}{E(-x)} = \frac{1}{E(|x|)} > 0.$$

Theorem Suppose that a and b are arbitrary reals. Then

$$E(a + b) = E(a)E(b)$$

Proof Define the function g by

$$g(x) = \frac{E(a + x)}{E(a)E(x)} = \frac{1}{E(a)} \cdot \frac{E(a + x)}{E(x)}$$

Since outputs of E are all positive g is defined and differentiable at all real x . We get

$$\begin{aligned} g'(x) &= \frac{1}{E(a)} \cdot \frac{E'(a + x)E(x) - E(a + x)E'(x)}{[E(x)]^2} \\ &= \frac{1}{E(a)} \cdot \frac{E(a + x)E(x) - E(a + x)E(x)}{[E(x)]^2} = 0 \end{aligned}$$

for all x . So $g(x)$ must be constant.

$$g(x) = g(0) = \frac{E(a)}{E(a)E(0)} = 1 \quad \text{for all } x$$

and thus

$$\begin{aligned}\frac{E(a+x)}{E(a)E(x)} &= 1 \quad \text{for all } x \\ E(a+x) &= E(a)E(x) \quad \text{for all } x \\ E(a+b) &= E(a)E(b).\end{aligned}$$

Theorem For all n in \mathbb{N} and any real x

$$E(nx) = [E(x)]^n$$

Proof Use induction on n .

Theorem For all n in \mathbb{Z} and any real x

$$E(nx) = [E(x)]^n$$

Proof Use induction on n .

Theorem For all q in \mathbb{Q} and any real x

$$E(qx) = [E(x)]^q$$

Proof Consider an arbitrary real x .

Write $q = k/\ell$ where we keep $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$.

Case $k = 1$. By the definition of the notation

$$[E(x)]^{1/\ell} \text{ is the non-negative } \ell^{\text{th}} \text{ root of } [E(x)].$$

But

$$\left[E\left(\frac{1}{\ell}x\right) \right]^\ell = E\left(\ell \cdot \frac{1}{\ell}x\right) = E(x).$$

Thus $E(\frac{1}{\ell}x)$ must be our unique ℓ^{th} root of $E(x)$ – or in other words

$$E\left(\frac{1}{\ell}x\right) = [E(x)]^{1/\ell}$$

General case

$$\begin{aligned}E\left(\frac{k}{\ell}x\right) &= E\left(k \cdot \frac{x}{\ell}\right) = \left[E\left(\frac{x}{\ell}\right) \right]^k = \left[E\left(\frac{1}{\ell}x\right) \right]^k \\ &= \left[(E(x))^{1/\ell} \right]^k = [E(x)]^{\frac{1}{\ell} \cdot k} = [E(x)]^{\frac{k}{\ell}}.\end{aligned}$$

1.7 Defining e and Understanding the Expression e^x

Definition We use the letter e to denote the output $E(1)$. In other words

$$e = E(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Homework By induction on n , verify that for all integers n

$$e^n = E(n)$$

Homework Verify that for all rational numbers q

$$e^q = E(q)$$

Definition For all irrational x

$$e^x = E(x)$$

Homework

In many books the following kind of recipe is given for understanding e^π :

"Take a sequence $(q_n)_{n=1}^\infty$ of rational numbers approximating π .

The sequence $(e^{q_n})_1^\infty$ will then approximate the number we mean by the notation e^π ."

- (a) Translate this recipe into more rigorous mathematics.
- (b) Explain why this recipe gives a well-defined value for e^π – in other words, why the recipe does not give different values for different sequences approaching π .

1.8 The natural logarithm

Theorem The function E is one-to-one and maps \mathbb{R} onto \mathbb{R}^+

Proof We have seen that the derivative of E is strictly positive on \mathbb{R} . This tells us that E is strictly increasing and thus one-to-one. It remains to show that every positive real is in the range of E .

It is easy to verify that,

$$\text{for all positive integers } n, \quad n < 2^n < e^n \quad \text{and} \quad e^{-n} < 2^{-n} < \frac{1}{n}$$

Consider an arbitrary positive real p . We can find positive integers j and k so that

$$\frac{1}{j} < p < k.$$

and thus

$$e^{-j} < p < e^k$$

Since E continuous on the interval $[-j, k]$ and $E(-j) < p < E(k)$, the intermediate value theorem now gives us a real x such that $E(x) = p$.

Note that because E is one-to-one, this x that solves $E(x) = p$ is uniquely determined.

Definition

E is called the *exponential function*.

The inverse function for the exponential function is called the *natural logarithm function*.

For positive p , the "natural logarithm of p " is denoted $\ln(p)$.

$$\ln(p) = x \quad \text{if and only if} \quad e^x = p.$$

We have seen that the exponential function maps \mathbb{R} one-to-one onto \mathbb{R}^+ . Thus the natural logarithm maps \mathbb{R}^+ one-to-one onto \mathbb{R} .

Proposition The natural logarithm function is differentiable everywhere on its domain and

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}.$$

Proof (method #1: Use the inverse function theorem). Since the natural logarithm is the inverse of E the inverse function theorem tells us that the natural logarithm is differentiable everywhere on its domain and that

$$\frac{d}{dx}[\ln(x)] = \frac{1}{E'(\ln(x))} = \frac{1}{E(\ln(x))} = \frac{1}{x}.$$

Proof (method #2: Use the definition of derivative and the properties of derivatives) Consider an arbitrary positive p . For all positive x which are different from p

$$\frac{\ln(x) - \ln(p)}{x - p} = \frac{1}{\frac{x-p}{\ln(x) - \ln(p)}} = \frac{1}{\frac{E(\ln(x)) - E(\ln(p))}{\ln(x) - \ln(p)}}.$$

Now take an arbitrary sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R} - \{p\}$ which converges to p . For each n set $y_n = \ln(x_n)$. By the continuity of the natural logarithm at p we know that $\lim(y_n) = \ln(p)$. Thus the fraction in the bottom of last displayed equation evaluated at x_n gives us

$$\frac{E(\ln(x_n)) - E(\ln(p))}{\ln(x_n) - \ln(p)} = \frac{E(y_n) - E(\ln(p))}{y_n - \ln(p)}$$

Taking the limit as n runs through \mathbb{N} and using various properties of E at $\ln(p)$ we get

$$\lim \frac{E(y_n) - E(\ln(p))}{y_n - \ln(p)} = E'(\ln(p)) = E(\ln(p)) = p$$

Since $p \neq 0$ we now get

$$\lim \frac{\ln(x_n) - \ln(p)}{x_n - p} = \frac{1}{\lim \frac{E(y_n) - E(\ln(p))}{y_n - \ln(p)}} = \frac{1}{p}.$$

Now by the sequential criterion for existence of functional limits

$$\lim \frac{\ln(x) - \ln(p)}{x - p} = \frac{1}{p}.$$

Thus the natural logarithm is differentiable and its derivative at any positive p is $1/p$.

Proposition

$$\lim \left(1 + \frac{1}{n}\right)^n = e$$

Proof

The existence of this limit is asserted without proof in many precalculus and calculus books. The existence proof is beyond the scope of those courses.

Now we can not only make the existence proof more easily by using the tools we have developed, but we can evaluate the limit.

For each positive integer n

$$\ln \left[\left(1 + \frac{1}{n}\right)^n \right] = n \ln \left(1 + \frac{1}{n}\right) = \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \frac{\ln \left(1 + \frac{1}{n}\right) - \ln(0)}{\frac{1}{n} - 0}$$

We know that the natural logarithm has a derivative at zero and that this derivative is 1. Take the sequential limit above.

$$\lim \left(\ln \left[\left(1 + \frac{1}{n}\right)^n \right] \right) = \lim \left(\frac{\ln \left(1 + \frac{1}{n}\right) - \ln(0)}{\frac{1}{n} - 0} \right) = 1$$

Now recall that the exponential function is continuous everywhere, and in particular is continuous at 1. Using the sequential criterion for limits of continuous functions

$$\lim \left(E \left(\ln \left[\left(1 + \frac{1}{n}\right)^n \right] \right) \right) = E(1) = e$$

But E is the inverse of the natural logarithm so

$$E \left(\ln \left[\left(1 + \frac{1}{n}\right)^n \right] \right) = \left(1 + \frac{1}{n}\right)^n$$

and thus

$$\lim \left(1 + \frac{1}{n}\right)^n = \lim E \left(\ln \left[\left(1 + \frac{1}{n}\right)^n \right] \right) = e.$$

Definition For each positive p and each real x we define the expression p^x by

$$p^x = e^{x \ln p}$$

Note that this makes sense because

$$p = E(\ln(p)) = e^{\ln(p)}$$

and

$$\left(e^{\ln(p)}\right)^x = e^{\ln(p) \cdot x} = e^{x \cdot \ln(p)}.$$

Further note that this function is differentiable by the chain rule and that

$$\begin{aligned} \frac{d}{dx} \left[e^{x \cdot \ln(p)} \right] &= \frac{d}{dx} [E(x \cdot \ln(p))] = E'(x \cdot \ln(p)) \cdot \ln p = E(x \cdot \ln(p)) \ln p \\ &= \ln p \cdot e^{x \ln p} = \ln p \cdot p^x \end{aligned}$$

Proposition There is a unique positive p with the property that the curve $y = p^x$ crosses the y -axis with slope 1. This value of p is e .

Proof The slope of the tangent to $y = p^x$ at the point $(0, p^0)$ is $\ln p \cdot p^0$, which is just $\ln p$. Note that

$$\ln p = 1 \text{ if and only if } E(1) = p \text{ if and only if } e = p.$$