

Homework Sets # 9 , Math 311:02, Fall 2008
Sample Solutions

§4.1 #1 Task Consider the function defined by $f(x) = x^2$ and an arbitrary point (x_0, y_0) on the graph of f subject only to the condition $x_0 \neq 0$. Without using the notion of derivative find the equation of a straight line that intersects this graph only at (x_0, y_0) .

Result: $y = (2x_0)(x - x_0) + (x_0)^2$.

Work: Since (x_0, y_0) is on the graph of f we know $y_0 = (x_0)^2$. Thus, for each real m , the line $y = m(x - x_0) + (x_0)^2$ intersects the graph of f at (x_0, y_0) . We need to find m so that the line cannot intersect the graph of f anywhere else. So we look for m such that the quadratic equation

$$x^2 - (m(x - x_0) + (x_0)^2) = 0$$

has only the one (repeated) real solution x_0 . Put this quadratic equation into standard form

$$x^2 + [-m]x + [mx_0 - (x_0)^2] = 0$$

The quadratic equation gives solutions

$$\frac{-[-m] \pm \sqrt{[-m]^2 - 4 \cdot 1 \cdot [mx_0 - (x_0)^2]}}{2 \cdot 1}$$

To get a repeated root x_0 we need

$$x_0 = \frac{m}{2} \quad \text{and} \quad m^2 - 4mx_0 + 4(x_0)^2 = 0$$

or equivalently

$$m = 2x_0 \quad \text{and} \quad (m - 2x_0)^2 = 0$$

which reduces to the condition we knew we needed namely, $m = 2x_0$.

§4.1 #2 Task Suppose that

f is a function; c is an accumulation point of $\text{dom}(f)$; $c \in \text{dom}(f)$; $L \in \mathbb{R}$

Prove the equivalence of the statements

$$(1) \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = L \quad \text{and} \quad (2) \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = L$$

Proof Define the function Q by

$$Q(h) = \frac{f(c+h) - f(c)}{h}$$

and note that

$$\text{Dom}(Q) = \{h : c+h \in \text{Dom}(f) \text{ and } h \neq 0\}$$

Step 1 Show that (1) implies (2)

Assume (1). Since c is an accumulation point of $\text{Dom}(f)$ we get a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{Dom}(f) - \{c\}$ such that $\lim(x_n) = c$. For each n define $h_n = x_n - c$. Note that for all n , $h_n \in \text{Dom}(Q) - \{0\}$ and $\lim(h_n) = 0$. This verifies that 0 is an accumulation point of $\text{Dom}(Q)$.

Now we can use the sequential criterion for functional convergence to verify the limit statement (2). Consider an arbitrary sequence $(z_n)_{n \in \mathbb{N}}$ in $\text{Dom}(Q) - \{0\}$ and assume that $\lim(z_n) = 0$. It is easy to check that the sequence $(c+z_n)_{n \in \mathbb{N}}$ is a sequence in $\text{Dom}(f) - \{c\}$ and that $\lim(c+z_n) = c$. By (1) and some arithmetic in the bottom of the fractions, we know that

$$\lim \frac{f(c+z_n) - f(c)}{z_n} = \lim \frac{f(c+z_n) - f(c)}{(c+z_n) - c} = L$$

This is just what we need to conclude that

$$(2) \lim \frac{f(c+h) - f(c)}{h} = L$$

Step 2 Show that (2) implies (1)

Assume (2). Note that in this direction our hypothesis takes explicit care of the domain and accumulation point issues. We use an $\varepsilon\delta$ -proof to finish (1). Consider an arbitrary positive ε . Use (2) to pick a positive δ with the property that

$$\text{for all } h \text{ in } \text{Dom}(Q), \quad 0 < |h - 0| < \delta \Rightarrow |Q(h) - L| < \varepsilon.$$

Now consider an arbitrary x in $\text{Dom}(f)$ and assume that $0 < |x - c| < \delta$. We immediately get

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| = |Q(x - c) - L| < \varepsilon.$$

So (1) is verified.

§4.1 #4 Task Let $g(x) = x^2$. Consider arbitrary real c . Use the definition of derivative to compute $g'(c)$.
Proof There is no problem with verifying that c is in the domain of g and that c is an accumulation point of the domain of g . Now compute for $x \neq c$.

$$\frac{g(x) - g(c)}{x - c} = \frac{x^2 - c^2}{x - c} = \frac{x - c}{x - c} \frac{x + c}{1} = x + c \rightarrow 2c \quad \text{as } x \rightarrow c.$$

§4.1 #5 Task Set

$$h(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use the computational rules of derivatives from Calc I and II.

- Find the derivative of h
- Show that h' is continuous on \mathbb{R}
- Show that there is one real c such that h' is differentiable on $\mathbb{R} - \{c\}$ but not at c .

Result (a)

$$h'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Work for (a) Using the algebra of derivatives we get $h'(c)$ easily for non-zero c

$$h'(c) = 3c^2 \sin\left(\frac{1}{c}\right) - c^3 \cos\left(\frac{1}{c}\right)c^{-2}$$

For $c = 0$ it is convenient to use the alternate definition

$$h'(0) = \lim_{t \rightarrow 0} \frac{h(0+t) - h(0)}{t} = \lim_{t \rightarrow 0} t^2 \sin\left(\frac{1}{t}\right) = 0$$

since the factor t^2 has limit 0 and the second factor is bounded.

Proof for (b) Freshman calculus gives us a derivative for h' at every non-zero c . So h' must be continuous at every non-zero c . To see that h' is continuous at 0 note that

$$\begin{aligned} h'(x) - h'(0) &= h'(x) = x [3x \sin(1/x)] - \cos(1/x) \\ |h'(x) - h'(0)| &= |x| \cdot |3x \sin(1/x) - \cos(1/x)| \leq |x - 0| \cdot (3|x| + 1) \end{aligned}$$

Thus

$$0 \leq \lim_{x \rightarrow 0} |h'(x) - h'(0)| \leq \lim_{x \rightarrow 0} |x - 0| \cdot (3|x| + 1) = 0$$

and we get the continuity at zero by the Squeeze Theorem.

Result and Proof for (c) As noted above h' will be differentiable at every non-zero c , where

$$\begin{aligned} h'(c) &= 3c^2 \sin\left(\frac{1}{c}\right) - c^3 \cos\left(\frac{1}{c}\right)c^{-2} = 3c^2 \sin\left(\frac{1}{c}\right) - c \cos\left(\frac{1}{c}\right) \\ h''(c) &= [6c \sin(1/c) - 3c^2 \cos(1/c)c^{-2}] - [\cos(1/c) - c \sin(1/c)c^{-2}] \\ &= [6c + c^{-1}] \sin(1/c) - [3 - 1] \cos(1/c) \end{aligned}$$

We must try to take a limit of the difference quotient at zero explicitly

$$\begin{aligned} \frac{h'(x) - h'(0)}{x - 0} &= \frac{(x [3x \sin(1/x)] - \cos(1/x)) - 0}{x} \\ &= \left(3x \sin(1/x) - \frac{\cos(1/x)}{x} \right) \end{aligned}$$

The first term converges to 0 as x runs to 0. But the second term is not bounded in any neighborhood of 0 so the second term cannot converge. If $h''(0)$ exists then the second term must converge. So $h''(0)$ cannot exist.

§4.1 #7 Task Suppose that f is defined on (a, b) . We say that

" f satisfies a Lipschitz condition at c in (a, b) "

iff

there is a positive M and a positive r such that

$$\text{for all } x \text{ in } (a, b) \quad |x - c| < r \Rightarrow |f(x) - f(c)| < M|x - c|$$

NOTE that I have translated this definition so that it agrees with the definition I gave earlier for "local Lipschitz condition".

- a) Give an example of a function that is continuous at c but fails to satisfy a Lipschitz condition at c
- b) Show that if a function is differentiable at c then it does satisfy a Lipschitz condition.

a) In light of (b) we look for a continuous function that fails to have a derivative at c . The absolute value function is tempting, but does satisfy a Lipschitz condition. So try $f(x) = x^{1/3}$ on the interval $(a, b) = (-1, 1)$. Since f is the inverse of a strictly increasing function, it must be continuous everywhere.

Suppose, for the sake of contradiction, that f does satisfy a Lipschitz condition at 0 with M and r as in the definition. Then for all x near 0 but not equal to 0

$$M > \left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \frac{x^{1/3} - 0}{x - 0} \right| = \left| \frac{1}{x^{2/3}} \right|$$

Now look at the sequence $(x_n)_{n \in \mathbb{N}}$ defined by

$$x_n = \frac{r}{2} \frac{1}{n^{3/2}} = \frac{r}{2} n^{-3/2}$$

Since we have for all indices

$$-1 < 0 < x_n < \frac{r}{2} \frac{1}{n^1} < \min(r, 1).$$

So for all indices

$$M > \left| \frac{1}{(x_n)^{2/3}} \right| = \left| \frac{1}{\left(\frac{r}{2} \cdot n^{-3/2}\right)^{2/3}} \right| = \left(\frac{r}{2}\right)^{-2/3} \cdot n$$

which says that $(r/2)^{2/3} M$ is an upper bound for \mathbb{N} , which is false.

§4.1 #11 Task Define f on $(0, 1)$ by

$$f(x) = \sqrt{2x^2 - 3x + 6}$$

show that f is differentiable at every point of its domain. Find the derivative.

Proof Let r denote the square root function, $r(u) = \sqrt{u}$. We know that r is defined on $[0, +\infty)$ and is differentiable on $(0, +\infty)$ with $r'(x) = x^{-1/2}$ for positive x .

Let P denote the polynomial $P(x) = 2x^2 - 3x + 6$. We know that P is continuous and differentiable everywhere.

Note that $f = r \circ P$. We check to see if we can apply the Chain Rule.

Suppose c is in $(0, 1)$. Then c is both in the domain of f and is an accumulation point for the domain of f . Our "inside function" P is differentiable at c . Our "outside function" r is differentiable at c provided that $P(c) > 0$. Note that

$$\begin{aligned} P(c) &= 2c^2 - 3c + 6 \\ &> -3c + 6 \quad \text{since } c > 0 \text{ and thus } 2c^2 > 0 \\ &> 6 - 3 = 3 > 0 \quad \text{since } c < 1 \text{ and thus } -3c > -3 \end{aligned}$$

So the Chain Rule Theorem tells us that f' exists and that

$$f'(c) = r'(P(c))P'(c) = \frac{1}{2} (2c^2 - 3c + 6) (4c - 3)$$

§4.1 #13 Task Suppose that function f maps $[a, b]$ into $[c, d]$ and function g maps $[c, d]$ into \mathbb{R} . Suppose further that f and f' are differentiable on $[a, b]$ and that g and g' are differentiable on $[c, d]$. Show that $g \circ f$ is differentiable, that $(g \circ f)'$ is also differentiable and compute the derivatives.

Results and Proof The hypotheses are sufficient to satisfy the hypotheses of the Chain Rule Theorem applied to $g \circ f$. Thus on $[a, b]$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

Next we need to see why $(g \circ f)'$ has a derivative. First note that $(g \circ f)'$ is a product of two functions. To apply the product rule we need to check that each factor is differentiable.

By hypotheses the second factor f' is differentiable. The first factor is the composition of g' following f . We are told that f maps $[a, b]$ differentiably into $[c, d]$ and that g' maps $[c, d]$ differentiably into \mathbb{R} . So by the Chain Rule Theorem, $(g' \circ f)$ is differentiable and

$$(g' \circ f)'(x) = g''(f(x)) \cdot f'(x)$$

Now we know that we can apply the product rule to get

$$\begin{aligned} (g \circ f)''(x) &= \left[(g' \circ f)'(x) \right] \cdot f'(x) + (g' \circ f)(x) \cdot [f''(x)] \\ &\quad [g''(f(x)) \cdot f'(x)] \cdot f'(x) + g'(f(x)) \cdot [f''(x)] \end{aligned}$$

§4.1 #14 Task Suppose that f is differentiable on \mathbb{R} and that g is defined by

$$g(x) = x^2 f(x^3)$$

Show that g is differentiable and find $g'(x)$.

Result $g'(x) = 2x f(x^3) + 3x^4 f'(x^3)$.

Proof. We use our usual notation for power functions

$$P_n(x) = x^n.$$

Then we see that g is the product of the function P_2 and the composition $f \circ P_3$. Since f and P_3 are both differentiable on \mathbb{R} we get

$$(f \circ P_3)'(x) = f'(P_3(x)) \cdot P_3'(x) = f'(x^3) \cdot 3x^2$$

We also know that

$$P_2'(x) = 2x$$

Thus

$$\begin{aligned} g'(x) &= P_2'(x) \cdot (f \circ P_3)(x) + P_2(x) \cdot (f \circ P_3)'(x) \\ &= 2x \cdot f(x^3) + x^2 \cdot [f'(x^3) \cdot 3x^2] \\ &= 2x \cdot f(x^3) + 3x^4 \cdot f'(x^3) \end{aligned}$$