640:151 Calculus I, Solutions to Practice Midterm Exam #2

1. Compute the following:

   (a) \( \int (7t^4 - t^{-2}) \, dt \)

   Solution.

   \[
   \int (7t^4 - t^{-2}) \, dt = 7\frac{t^5}{5} - \frac{1}{t} + C = \frac{7}{5}t^5 + \frac{1}{t} + C.
   \]

   (b) \( \int \left( \frac{2}{5} \sin x - 10 \cos x \right) \, dx \)

   Solution.

   \[
   \int \left( \frac{2}{5} \sin x - 10 \cos x \right) \, dx = -\frac{2}{5} \cos x - 10 \sin x + C.
   \]
1. (continued) Compute the following:

(c) \( \int (\sec(2x + 3) \tan(2x + 3)) \, dx \)

Solution.

\[
\int (\sec(2x + 3) \tan(2x + 3)) \, dx = \int \frac{\sin(2x + 3)}{\cos^2(2x + 3)} \, dx = \\
= \int [\cos(2x + 3)]^{-2} [\cos(2x + 3)]' \cdot \left( \frac{-1}{2} \right) \, dx = \\
= [\cos(2x + 3)]^{-1} \cdot (-1) \cdot \left( \frac{-1}{2} \right) + C = \\
= \frac{1}{2 \cos(2x + 3)} + C.
\]
1. (continued) Compute the following:

(d) \( (xe^{x^2 + \cos x})' \)

Solution.

Note that
\[
x e^{x^2 + \cos x} = e^{\ln(x e^{x^2 + \cos x})} = e^{(e^{x^2 + \cos x}) \ln x}.
\]

It follows that
\[
(x e^{x^2 + \cos x})' = e^{(e^{x^2 + \cos x}) \ln x} \left[ \left( e^{x^2 + \cos x} \right)' \ln x + (e^{x^2 + \cos x}) \left( \ln x \right)' \right] =
\]
\[
= e^{(e^{x^2 + \cos x}) \ln x} \left[ (2xe^{x^2} - \sin x) \ln x + (e^{x^2 + \cos x}) \frac{1}{x} \right] =
\]
\[
= xe^{x^2 + \cos x} \left[ e^{x^2} \left( 2x \ln x + \frac{1}{x} \right) - (\sin x)(\ln x) + \frac{\cos x}{x} \right].
\]

(e) \( \frac{d}{dx} \log_{11}(\sin x) \)

Solution.

Since \( \log_{11}(\sin x) = \frac{\ln(\sin x)}{\ln 11} \),
\[
\frac{d}{dx} \log_{11}(\sin x) = \frac{d}{dx} \left( \frac{\ln(\sin x)}{\ln 11} \right) = \frac{1}{\ln 11} \frac{\cos x}{\sin x} = \frac{\cot x}{\ln 11}.
\]
1. (continued) Compute the following:

\[
(f) \quad \frac{d}{dx} \sec^{-1} (2e^x + 3) \bigg|_{x=0}
\]

Solution.

\[
\frac{d}{dx} \sec^{-1} (2e^x + 3) \bigg|_{x=0} = \frac{(2e^x + 3)'}{(2e^x + 3)\sqrt{(2e^x + 3)^2 - 1}} \bigg|_{x=0} =
\]

\[
= \frac{2e^x}{(2e^x + 3)\sqrt{(2e^x + 3)^2 - 1}} \bigg|_{x=0} =
\]

\[
= \frac{2}{5\sqrt{5^2 - 1}} = \frac{1}{5\sqrt{6}}.
\]
2. Consider the function \( f(x) = xe^{-2x} \).

(a) Find the intervals on which the function \( f \) is increasing or decreasing.

Solution.

Since \( f'(x) = e^{-2x}(1 - 2x) \),

\[
f'(x) > 0 \iff x < \frac{1}{2} \quad \text{and} \quad f'(x) < 0 \iff x > \frac{1}{2}.
\]

Therefore, \( f \) is increasing on \(( -\infty, \frac{1}{2} )\) and \( f \) is decreasing on \([ \frac{1}{2}, \infty )\).

(b) Determine the intervals on which the function \( f \) is concave up or down and find the points of inflection.

Solution.

Using \( f'(x) = e^{-2x}(1 - 2x) \), we obtain that

\[
f''(x) = e^{-2x}(-2)(1 - 2x) + e^{-2x}(-2) = e^{-2x}(-2)(2 - 2x) = 4(x - 1)e^{-2x}.
\]

It follows that

\[
f''(x) > 0 \iff x > 1, \quad f''(x) < 0 \iff x < 1.
\]

Therefore, \( f \) is concave up on \((1, \infty)\), \( y \) is concave down on \(( -\infty, 1)\), and \( f \) has an inflection point at \( x = 1 \).
2. (continued) Consider the function \( f(x) = xe^{-2x} \).

(c) Does \( f \) have a minimum on the interval \([0, 2]\)? If yes, explain why and find it. If not, explain why.

Solution.

\[
x e^{-2x} \geq f(0) = 0 \quad \text{for all} \quad x \in [0, 2].
\]

It follows that 0 is the minimum value of \( f \) on \([0, 2]\).

(d) Does \( f \) have a maximum on the interval \([0, 2]\)? If yes, explain why and find it. If not, explain why.

Solution.

By part (a),

\[
f(x) \leq f \left( \frac{1}{2} \right) \quad \text{for all} \quad x \in [0, 2].
\]

Therefore, \( f \left( \frac{1}{2} \right) = \frac{1}{2e} \) is the maximum value of \( f \) on \([0, 2]\).
2. (continued) Consider the function \( f(x) = xe^{-2x} \).

(e) Find all vertical and horizontal asymptotes to the graph of \( f \).

\textit{Solution.}

Since \( f \) is continuous on \( \mathbb{R} \), the graph of \( f \) has no vertical asymptotes. Note that

\[
\lim_{x \to -\infty} f(x) = -\infty,
\]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{2e^{2x}} = 0.
\]

For the limit at \( \infty \) above, we have used L'Hôpital’s Rule. It follows that \( y=0 \) is the only horizontal asymptote to the graph of \( f \).

(f) Sketch the graph of \( f \).

\textit{Solution.}

Note first that the graph of \( f \) intersects the coordinate axes at the origin only. By part (a), \( f \left( \frac{1}{2} \right) = \frac{1}{2e} \) is the absolute maximum of \( f \).

See next page for the graph.
absolute max \( \frac{1}{2e} \)

\( \frac{1}{e^2} \) = point of inflection

\( y = xe^{-x} \)
3. Determine whether the following limit exists. If it exists, compute it. If it does not exist, say so and explain why.

(a) \( \lim_{x \to 0} \frac{\sin^{-1} x}{\tan^{-1} x} \).

Note: \( \sin^{-1} = \arcsin \), \( \tan^{-1} = \arctan \).

Solution.
Using L'Hôpital’s Rule,

\[
\lim_{x \to 0} \frac{\sin^{-1} x}{\tan^{-1} x} = \lim_{x \to 0} \frac{\left(\sin^{-1} x\right)'}{\left(\tan^{-1} x\right)'} = \lim_{x \to 0} \frac{1}{\sqrt{1-x^2}} \frac{1}{1+x^2} = 1.
\]
3. \(\text{(continued)}\) Determine whether the following limit exists. If it exists, compute it. If it does not exist, say so and explain why.

(b) \(\lim_{x \to 0} \left( \frac{1}{x^2} - \cot^2 x \right)\).

Solution.

\[
\lim_{x \to 0} \left( \frac{1}{x^2} - \cot^2 x \right) = \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{\sin^2 x - x^2 (1 - \sin^2 x)}{x^2 \sin^2 x} = \\
= \lim_{x \to 0} \frac{\sin^2 x - x^2 + x^2 \sin^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{(1 + x^2) \sin^2 x - x^2}{x^2 \sin^2 x} = \\
= \lim_{x \to 0} \frac{(1 + x^2) \sin^2 x - x^2}{x^4} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \frac{(1 + x^2) \sin^2 x - x^2}{x^4} = \\
= \lim_{x \to 0} \frac{(1 + x^2) \sin^2 x - x^2}{x^4} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 x + (1 + x^2) 2 \sin x \cos x - 2 x}{4 x^3} = \\
= \lim_{x \to 0} \frac{2 x (\sin^2 x - 1) + (1 + x^2) \sin(2x)}{4 x^3} = \lim_{x \to 0} \frac{-2 x \cos^2 x + (1 + x^2) \sin(2x)}{4 x^3} = \\
= \lim_{x \to 0} \frac{-2 \cos^2 x + 4 x \sin(2x) + (1 + x^2) 2 \cos(2x)}{12 x^2} = \lim_{x \to 0} \frac{-2 \cos^2 x + 4 x \sin(2x) + (1 + x^2) 2 \cos(2x)}{12 x^2} = \\
= \lim_{x \to 0} \frac{-2 \cos^2 x + 4 x \sin(2x) + 4 x^2 \cos^2 x + 4 \cos^2 x - 2 - 2 x}{12 x^2} = \\
= \lim_{x \to 0} \frac{2 \cos^2 x - 2 + 4 x \sin(2x) + 4 x^2 \cos^2 x - 2 x}{12 x^2} = \\
= \lim_{x \to 0} \frac{-2 \sin^2 x + 4 \sin(2x)}{x^2} + 4 \cos^2 x - 2} = \frac{2}{3}.

\]

We used L'Hôpital’s Rule twice. We also used the fact that \(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\) for the middle equality on line 3 and twice for the last equality.
4. Find an equation of the tangent line to the curve \( \sin(x - y) = x \cos\left(y + \frac{\pi}{4}\right) \) at the point \( \left(\frac{\pi}{4}, \frac{\pi}{4}\right) \).

Solution.

On the given curve, 

\[
\frac{d}{dx} \sin(x - y) = \frac{d}{dx} \left[ x \cos\left(y + \frac{\pi}{4}\right) \right]
\]

and so

\[
(1 - y') \cos(x - y) = \cos\left(y + \frac{\pi}{4}\right) - xy' \sin\left(y + \frac{\pi}{4}\right) \iff
y' \left[ -\cos(x - y) + x \sin\left(y + \frac{\pi}{4}\right) \right] = -\cos(x - y) + \cos\left(y + \frac{\pi}{4}\right).
\]

Taking \( x = \frac{\pi}{4} \) and \( y = \frac{\pi}{4} \) in the above equation, we obtain that

\[
y' \bigg|_{\left(\frac{\pi}{4}, \frac{\pi}{4}\right)} \cdot \left( -1 + \frac{\pi}{4} \right) = -1.
\]

Thus, the slope of the required tangent line is

\[
y' \bigg|_{\left(\frac{\pi}{4}, \frac{\pi}{4}\right)} = \frac{4}{4 - \pi}
\]

and the required equation is

\[
y - \frac{\pi}{4} = \frac{4}{4 - \pi} \left(x - \frac{\pi}{4}\right).
\]
5. (a) At a given moment, a plane passes directly above a radar station at an altitude of 6 km. The plane’s speed is 800 km/h. Let \( \theta \) be the angle that the line through the radar station and the plane makes with the horizontal. How fast is \( \theta \) changing 12 min after the plane passes over the radar station?

**Note:** You do NOT have to simplify your final numerical answer.

**Solution.**

We know that, at a given moment \( t_0 \), the plane passes through the point \( O \) which is above the radar station. Let \( x(t) \) be the distance between the plane at time \( t \) and \( O \).

We are given that \( \frac{dx}{dt} = 800 \text{ km/h} \) and we want to find \( \frac{d\theta}{dt} \) when \( t = t_0 + 12 \text{ min} \).

Note that

\[
\cot \theta = \frac{x}{6 \text{ km}}.
\]

(1)

Differentiating (1) with respect to \( t \), we obtain that

\[
-\frac{d\theta}{dt} \frac{1}{\sin^2 \theta} = \frac{1}{6 \text{ km}} \frac{dx}{dt}.
\]

Therefore

\[
\frac{d\theta}{dt} = -\frac{1}{6 \text{ km}} \sin^2 \theta \frac{dx}{dt}.
\]

(2)

Since the plane travels 800 km in one hour, it will travel \( \frac{800 \text{ km}}{5} = 160 \text{ km} \) in 12 min.

So at time \( t = t_0 + 12 \text{ min} \), \( x = 160 \text{ km} \). Using this and (1), we obtain that, at time \( t = t_0 + 12 \text{ min} \)

\[
\cot \theta = \frac{160 \text{ km}}{6 \text{ km}} = \frac{80}{3}.
\]

It follows that, at time \( t = t_0 + 12 \text{ min} \),

\[
\sin^2 \theta = \frac{1}{1 + \cot^2 \theta} = \frac{1}{1 + \frac{6400}{9}} = \frac{9}{6409}.
\]

Using this and (2), we obtain that

\[
\left. \frac{d\theta}{dt} \right|_{t=t_0+12 \text{ min}} = -\frac{1}{6} \cdot \frac{9}{6409} \cdot 800 \frac{\text{rad}}{h}.
\]

**Note:** Omitting “rad” above is perfectly fine.
5. (continued)

(b) Consider an isosceles trapezoid with a base of length 4 and sides of length 2. Find the angles of the trapezoid that maximize the area of the trapezoid.

**Solution.**

Let $\theta$ be the angle of the trapezoid such that $\theta \leq \frac{\pi}{2}$. It follows that the height of the trapezoid is $2 \sin \theta$. There are next two cases to consider:

*Case 1.* the other base is $4 + 4 \cos \theta$.

*Case 2.* the other base is $4 - 4 \cos \theta$.

We first consider Case 1. Then the area of the trapezoid is

$$
(8 + 4 \cos \theta) \sin \theta = 8 \sin \theta + 2 \sin(2 \theta).
$$

Let $f: \left[0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ be defined by

$$
f(\theta) = 4 \sin \theta + \sin(2 \theta).
$$

Note that

$$
f'(\theta) = 4 \cos \theta + 2 \cos(2 \theta) = 4 \cos \theta + 2(2 \cos^2 \theta - 1) = 4 \cos^2 \theta + 4 \cos \theta - 2 = 2(2 \cos^2 \theta + 2 \cos \theta - 1).
$$

Note that

$$
2x^2 + 2x - 1 < 0 \iff x \in \left(\frac{-1 - \sqrt{3}}{2}, \frac{-1 + \sqrt{3}}{2}\right),
$$

$$
2x^2 + 2x - 1 > 0 \iff x \in \left(-\infty, \frac{-1 - \sqrt{3}}{2}\right) \cup \left(\frac{-1 + \sqrt{3}}{2}, \infty\right).
$$

Note that $0 < \frac{-1 + \sqrt{3}}{2} < 1$. Let

$$
\theta_0 \equiv \cos^{-1} \left(\frac{-1 + \sqrt{3}}{2}\right)
$$

By (5) and (4),

$$
f'(\theta) > 0 \text{ for all } \theta \in (0, \theta_0), \quad f'(\theta) < 0 \text{ for all } \theta \in \left(\theta_0, \frac{\pi}{2}\right).
$$

Therefore,

$$
f \text{ is increasing on } [0, \theta_0] \text{ and } f \text{ is decreasing on } \left[\theta_0, \frac{\pi}{2}\right].
$$

It follows that $f(\theta_0)$ is the maximum of $f$ on $[0, \frac{\pi}{2}]$ and this maximum is attained only at $\theta_0$. 


Note that
\[ \sin \theta_0 = \sqrt{1 - \cos^2 \theta_0} = \sqrt{1 - \frac{4 - 2\sqrt{3}}{4}} = \frac{\sqrt{3}}{2}. \]

Using (3), we obtain that the area of the trapezoid with angles \( \theta_0 \) and \( \pi - \theta_0 \) is
\[ \left[ 8 + 2 \left(-1 + \sqrt{3}\right) \right] \sqrt{\frac{\sqrt{3}}{2}}. \]  
(6)

We next consider Case 2. Then the area of the trapezoid is
\[ (8 - 4 \cos \theta) \sin \theta = 8 \sin \theta - 2 \sin(2\theta). \]

Let \( g: [0, \frac{\pi}{2}] \rightarrow \mathbb{R} \) be defined by
\[ g(\theta) = 4 \sin \theta - \sin(2\theta). \]

Note that
\[ g'(\theta) = 4 \cos \theta - 2 \cos(2\theta) = 4 \cos \theta - 2(2 \cos^2 \theta - 1) = -4 \cos^2 \theta + 4 \cos \theta + 2 = 2 (-2 \cos^2 \theta + 2 \cos \theta + 1). \]  
(7)

Note that
\[ -2x^2 + 2x - 1 > 0 \iff x \in \left(\frac{1 - \sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2}\right). \]  
(8)

Since
\[ \frac{1 - \sqrt{3}}{2} < 0 < 1 < \frac{1 + \sqrt{3}}{2}, \]
\[ g'(\theta) > 0 \quad \text{for all} \quad \theta \in \left(0, \frac{\pi}{2}\right). \]

Therefore, \( g \) is increasing on \( [0, \frac{\pi}{2}] \) and so its maximum value on \( [0, \frac{\pi}{2}] \) is \( g \left(\frac{\pi}{2}\right) \).

The area of the trapezoid with all angles \( \frac{\pi}{2} \) is
\[ 8. \]  
(9)

We next compare the two areas (6) and (9):
\[ 8 < \left[ 8 + 2 \left(-1 + \sqrt{3}\right) \right] \sqrt{\frac{\sqrt{3}}{2}} \iff 8 \sqrt{\frac{2}{\sqrt{3}}} < 8 + 2 \left(-1 + \sqrt{3}\right) \iff \]
\[ \iff 4 \sqrt{\frac{2}{\sqrt{3}}} < 3 + \sqrt{3} \iff 16 \cdot \frac{2}{\sqrt{3}} < 12 + 6\sqrt{3} \iff 32 < 12\sqrt{3} + 18 \iff \]
\[ \iff 14 < 12\sqrt{3} \iff 7 < 6\sqrt{3} \iff 49 < 36 \cdot 3. \]

Since the last inequality above holds, the first inequality holds as well and so the angles of the trapezoid that maximize the area of the trapezoid are
\[ \theta_0 \quad \text{and} \quad \pi - \theta_0. \]
6. State and prove Rolle’s Theorem.

*See p220 of the textbook.*