1. Compute the following derivatives:

(a) (4 points) \([(2x^3 + 3x + 1) \cos x]’

\[ \frac{d}{dx} [(2x^3 + 3x + 1) \cos x] = (2x^3 + 3x + 1)' \cos x + (2x^3 + 3x + 1) \cos x’ = (6x^2 + 3) \cos x - (2x^3 + 3x + 1) \sin x. \]

(b) (4 points) \(\left( \frac{\sin x}{x^2 + 2x + 5} \right)’

\[ \frac{d}{dx} \left( \frac{\sin x}{x^2 + 2x + 5} \right) = \frac{(\sin x)'(x^2 + 2x + 5) - (\sin x)(x^2 + 2x + 5)'}{(x^2 + 2x + 5)^2} = \frac{(\cos x)(x^2 + 2x + 5) - (\sin x)(2x + 2)}{(x^2 + 2x + 5)^2} \]
1. (continued) Compute the following derivatives:

(c) (5 points) \( \left( \sqrt{\frac{x}{3x+1}} \right)' \)

Solution.

\[
\left( \sqrt{\frac{x}{3x+1}} \right)' = \frac{\left( \frac{x}{3x+1} \right)'}{2 \sqrt{\frac{x}{3x+1}}},
\]

\[
\left( \frac{x}{3x+1} \right)' = \frac{3x + 1 - 3x}{(3x + 1)^2} = \frac{1}{(3x + 1)^2}.
\]

The last two lines imply that

\[
\left( \sqrt{\frac{x}{3x+1}} \right)' = \frac{1}{2(3x + 1)^2 \sqrt{\frac{x}{3x+1}}}.
\]

(d) (5 points) \([e^{-2x+5}(3x + 1001)]'\)

Solution.

\[
[e^{-2x+5}(3x + 1001)]' = (e^{-2x+5})'(3x + 1001) + e^{-2x+5}(3x + 1001)' =
\]

\[
e^{-2x+5}(-2)(3x + 1001) + e^{-2x+5} \cdot 3 =
\]

\[
e^{-2x+5} (-6x - 2002 + 3) =
\]

\[
e^{-2x+5} (-6x - 1999).
\]
2. (a) (5 points) Define continuity of a function at a point.
Answer.
Assume that $f$ is a function defined on an open interval containing $c$. The function $f$ is continuous at $x = c$ if and only if $\lim_{x \to c} f(x) = f(c)$.

(b) (2 point) Define continuity of a function on $\mathbb{R}$. (In other words, assume that $g: \mathbb{R} \to \mathbb{R}$ is a function. What does it mean that $g$ is continuous on $\mathbb{R}$? Your answer should be a rigorous mathematical definition.)
Answer.
A function $g: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if and only if $g$ is continuous at $x = c$ for all $c \in \mathbb{R}$. 
2. (continued)

(c) (4 points) The function \( h: \mathbb{R} \to \mathbb{R} \) is defined by

\[
  h(x) = \begin{cases} 
    Cx & \text{if } x < 3, \\
    -x^2 & \text{if } x \geq 3,
  \end{cases}
\]

where \( C \) is a constant (in other words, \( C \) is a fixed real number). Find all values of \( C \) that make \( h \) continuous on \( \mathbb{R} \).

\textit{Solution.}

The function \( h \) is continuous at \( x = x_0 \) whenever \( x_0 \neq 3 \) because \( Cx \) and \( -x^2 \) are continuous everywhere [since polynomial functions are continuous everywhere]. It follows that

\[ h \text{ is continuous on } \mathbb{R} \text{ if and only if } h \text{ is continuous at } x = 3. \]  

By definition,

\[ h \text{ is continuous at } x = 3 \text{ if and only if } \lim_{x \to 3} h(x) = h(3). \]  

On the other hand,

\[ \lim_{x \to 3} h(x) \text{ exists if and only if } \lim_{x \to 3^-} h(x) = \lim_{x \to 3^+} h(x). \]  

Using the continuity of \( Cx \) and \( -x^2 \), we find that

\[ \lim_{x \to 3^-} h(x) = \lim_{x \to 3^-} Cx = 3C, \]

\[ \lim_{x \to 3^+} h(x) = \lim_{x \to 3^+} -x^2 = -3^2. \]

By (3) and (4),

\[ \lim_{x \to 3} h(x) \text{ exists if and only if } 3C = -3^2 \text{ if and only if } C = -3. \]  

On the other hand, \( h(3) = -3^2 \) and so, using (5), (2), and (1), we conclude that \( h \) is continuous on \( \mathbb{R} \) if and only if \( C = -3 \).
3. Consider \( f(x) = \frac{-x^2 + 100}{x^2 - 2x + 1} \).

(a) (2 points) Find all numbers \( c \) such that \( f \) is continuous at \( x = c \).

Solution. The function \( f \) is rational and so it is continuous on its domain. Its domain is \( \{ x \in \mathbb{R} : x \neq 1 \} \). Therefore, \( h \) is continuous at \( x = c \) if and only if \( c \neq 1 \).
3. (continued) Consider \( f(x) = \frac{-x^2 + 100}{x^2 - 2x + 1} \).

(b) (3 points) Decide whether \( \lim_{x \to \infty} f(x) \) exists. If it exists, compute it.

Solution.
Whenever \( x \neq 0 \),
\[
f(x) = \frac{-x^2 + 100}{x^2 - 2x + 1} = \frac{-1 + \frac{100}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}.
\]

Since \( \lim_{x \to \infty} \frac{100}{x^2} = \lim_{x \to \infty} \frac{2}{x} = \lim_{x \to \infty} \frac{1}{x^2} = 0 \),
\[
\lim_{x \to \infty} f(x) = -1.
\]

(c) (3 points) Decide whether \( \lim_{x \to -\infty} f(x) \) exists. If it exists, compute it.

Solution.
Using (6) in part (b) above and using the fact that
\[
\lim_{x \to -\infty} \frac{100}{x^2} = \lim_{x \to -\infty} \frac{2}{x} = \lim_{x \to -\infty} \frac{1}{x^2} = 0,
\]
we obtain that
\[
\lim_{x \to -\infty} f(x) = -1.
\]
3. (continued) Consider \( f(x) = \frac{-x^2 + 100}{x^2 - 2x + 1} \).

(d) (2 points) Find an equation of each horizontal asymptote of the graph of \( f \). If there are none, say so and explain why.

Solution.
By parts (b) and (c) above, \([y = -1]\) is the only horizontal asymptote of the graph of \( f \).
4. (16 points) Prove that the equation \( \frac{x^2}{x^7 + 1} - 0.44411 = 0 \) has at least one solution.

**Solution 1.**

**Proof.** Let \( f: [0, 1] \rightarrow \mathbb{R} \) be the function defined by

\[
 f(x) = \frac{x^2}{x^7 + 1} - 0.44411.
\]

The function \( f \) is continuous on \([0, 1]\) since it is a rational function whose domain is the set of all real numbers \( x \) with \( x \neq -1 \).

We note that

\[
 f(0) = -0.44411 < 0, \\
 f(1) = \frac{1}{2} - 0.44411 > 0. 
\]

By (7), (8), and the Intermediate Value Theorem, there exists at least one number \( c \in (0, 1) \) such that \( f(c) = 0 \). 

**Solution 2.**

**Proof.** Let \( g: [0, \infty) \rightarrow \mathbb{R} \) be the function defined by

\[
 g(x) = \frac{x^2}{x^7 + 1} - 0.44411.
\]

The function \( g \) is continuous on \([0, \infty)\) since it is a rational function whose domain is the set of all real numbers \( x \) with \( x \neq -1 \).

We note that

\[
 \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x^2}{x^7 + 1} - 0.44411 = \lim_{x \to \infty} \frac{\frac{1}{x^5}}{1 + \frac{1}{x^7}} - 0.44411 = -0.44411 < 0, \\
 g(1) = \frac{1}{2} - 0.44411 > 0. 
\]

By (9), (10), and the Intermediate Value Theorem, there exists at least one number \( c > 1 \) such that \( g(c) = 0 \). 

\[8\]
5. For each of the following limits, either evaluate or state why the limit does not exist. Show and justify all steps.

(a) (4 points)
\[
\lim_{x \to 0} \sin(5\pi x)
\]

Solution.
The function \( \sin(5\pi x) \) is continuous since it is the composition of two continuous functions: \( \sin x \) and \( 5\pi x \). Therefore,
\[
\lim_{x \to 0} \sin(5\pi x) = \sin(5\pi \cdot 0) = \sin 0 = 0
\]

(b) (6 points)
\[
\lim_{x \to 0} \frac{\tan(2x)}{\sin(5\pi x)}
\]

Solution.
We note that
\[
\frac{\tan(2x)}{\sin(5\pi x)} = \frac{\sin(2x)}{\cos(2x) \sin(5\pi x)} = \frac{\sin(2x)}{2x} \cdot \frac{2x}{\cos(2x)} \cdot \frac{5\pi x}{5\pi x} \cdot \frac{1}{\sin(5\pi x)} = \frac{\sin(2x)}{2x} \cdot \frac{5\pi x}{5\pi x} \cdot \frac{2}{\cos(2x)} \cdot \frac{1}{\sin(5\pi x)}
\]
whenever \( x \in \left(-\frac{1}{5}, 0\right) \cup \left(0, \frac{1}{5}\right) \). Since \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) and since \( \lim_{x \to 0} \cos(2x) = 1 \),
\[
\lim_{x \to 0} \frac{\tan(2x)}{\sin(5\pi x)} = \frac{2}{5\pi}
\]
5. (continued) For the following limit, either evaluate or state why the limit does not exist. Show and justify all steps.

(c) (10 points)

\[ \lim_{x \to 0} 2x^2 \sin \left( \frac{1}{x} \right) \]

**Solution.**

We note that

\[-1 \leq \sin \left( \frac{1}{x} \right) \leq 1 \quad \text{(11)}\]

whenever \( x \neq 0 \). By (11),

\[-2x^2 \leq 2x^2 \sin \left( \frac{1}{x} \right) \leq 2x^2 \quad \text{(12)}\]

whenever \( x \neq 0 \). Since polynomial functions are continuous everywhere,

\[ \lim_{x \to 0} (-2x^2) = \lim_{x \to 0} (2x^2) = 0. \quad \text{(13)}\]

By (12), (13), and the Squeeze Theorem,

\[ \lim_{x \to 0} 2x^2 \sin \left( \frac{1}{x} \right) = 0. \]
6. (25 points) Assume that
\[ \lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \]
where \( c, L, \) and \( M \) are real numbers. Prove that
\[ \lim_{x \to c} (f(x) + g(x)) = L + M. \]

The proof is shown in the textbook on pages 114, 115. If you like, here is another presentation.

Proof. Let \( \epsilon > 0 \). We will prove that

there exists \( \delta > 0 \) such that \( |f(x) + g(x) - (L + M)| < \epsilon \) whenever \( 0 < |x - c| < \delta. \) (14)

Since \( \lim_{x \to c} f(x) = L \), there exists \( \delta_1 > 0 \) such that
\[ |f(x) - L| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \] (15)

Since \( \lim_{x \to c} g(x) = M \), there exists \( \delta_2 > 0 \) such that
\[ |g(x) - M| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_2. \] (16)

By the triangle inequality,
\[ |f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|. \] (17)

By (15), (16), and (17), the claim (14) holds with \( \delta = \min\{\delta_1, \delta_2\}. \)