Some of the problems on the second midterm will be chosen from the following list [up to a change in notation]. The actual exam problems may have multiple parts and be a combination of some of the questions below.

We denote by $\mathbb{R}$ the set of real numbers. [This is denoted $\mathcal{R}$ in the textbook.]

1. Define the span of a finite nonempty set of vectors in $\mathbb{R}^n$. Give an example and draw a picture which illustrates your example. [p66]

2. Let $S = \{u_1, u_2, \ldots, u_k\}$ be a set of vectors from $\mathbb{R}^n$, and let $v$ be a vector in $\mathbb{R}^n$. Prove that

$$\text{Span}\{u_1, u_2, \ldots, u_k, v\} = \text{Span}S \iff v \in \text{Span}S.$$  

[p71-72]

3. Define linear dependence and linear independence of a finite set of vectors in $\mathbb{R}^n$. Give an example for each of the two concepts. Draw a picture to illustrate linear dependence. [p75, 76, and lecture notes]

4. Let $u_1, u_2, \ldots, u_k$ be vectors in $\mathbb{R}^n$. Prove that $u_1, u_2, \ldots, u_k$ are linearly dependent if and only if $u_1 = 0$ or there exists an $i \geq 2$ such that $u_i$ is a linear combination of the preceding vectors $u_1, u_2, \ldots, u_{i-1}$. [p81 and lecture notes]

5. Prove that the span of a finite nonempty subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. [p231]

6. Prove that the null space of a matrix is a subspace of some $\mathbb{R}^n$. [p233]

7. Let $S$ be a finite generating set for a nonzero subspace $V$ of $\mathbb{R}^n$. Prove that $S$ can be reduced to a basis for $V$ by removing vectors from $S$. [see p243 and your notes from the lecture]

8. Let $S$ be a linearly independent subset of a nonzero subspace $V$ of $\mathbb{R}^n$. Prove that $S$ can be extended to a basis for $V$ by inclusion of additional vectors. Prove that every nonzero subspace has a basis. [p245]

Note: Your proof should include the proof of property 4 on p81 which is given on p82.

9. Let $V$ be a nonzero subspace of $\mathbb{R}^n$. Prove that any two bases for $V$ contain the same number of vectors. [p245]

10. Let $V$ be a subspace of $\mathbb{R}^n$ with dimension $k$. Prove that every linearly independent subset of $V$ contains at most $k$ vectors. [p246]

11. Let $V$ be a $k$-dimensional subspace of $\mathbb{R}^n$. Prove that a set of $k$ linearly independent vectors from $V$ is a basis for $V$.

Prove that a generating set for $V$ which consists of $k$ vectors is a basis for $V$. [p248]

12. Prove that a set of eigenvectors of a square matrix that correspond to distinct eigenvalues is linearly independent. [p317-318]

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When preparing these problems for the midterm you may want to consider not only the indicated pages in the textbook, but also your notes from the lectures. In some cases you might find the presentation in class easier to follow.