Your full name in capital letters:

ID number:

Practice Midterm Exam # 2

DO NOT OPEN THIS EXAM OR BEGIN WRITING until the course instructor has announced the beginning of the examination

- No calculators. No cell phones. No questions. No textbooks or additional materials.
- Any statement should be supported by a proof or by a clear citation of a theorem/definition. All steps of a computation should be clearly indicated and justified. Answers without justification/work may receive zero credit.
- Hand in the exam no later than 2:30pm.
- Use blue or black ballpoint pens please. Answers written in pencil will be disregarded.
- Write neatly. Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want marked.
- Incorrect statements will be penalized. If you do not want something graded, please cross it out.

NOTE: QUESTIONS THAT ARE NOT ON THIS PRACTICE MIDTERM MAY BE ON THE ACTUAL MIDTERM. MOST IMPORTANTLY, TOPICS THAT ARE NOT COVERED ON THIS PRACTICE MIDTERM, BUT ARE PART OF THE SYLLABUS, MAY BE ON THE ACTUAL MIDTERM.
1. Let $S$ be a finite generating set for a nonzero subspace $V$ of $\mathbb{R}^n$. Prove that $S$ can be reduced to a basis for $V$ by removing vectors from $S$. 

2. Let $V$ be a nonzero subspace of $\mathbb{R}^n$. Prove that any two bases for $V$ contain the same number of vectors.
3. (a) Compute the determinant of the matrix
\[
\begin{bmatrix}
1 & -1 & 2 & 1 \\
2 & -1 & -1 & 4 \\
-4 & 5 & -10 & -6 \\
3 & -2 & 10 & -1
\end{bmatrix}.
\]

(b) Determine whether the vectors
\[
\begin{bmatrix}
1 \\
2 \\
-4 \\
3
\end{bmatrix}, \begin{bmatrix}
-1 \\
-1 \\
5 \\
-2
\end{bmatrix}, \begin{bmatrix}
2 \\
-1 \\
-10 \\
10
\end{bmatrix}, \begin{bmatrix}
1 \\
4 \\
-6 \\
-1
\end{bmatrix}
\]

are linearly independent.

(c) Determine whether the vectors
\[
\begin{bmatrix}
1 \\
-1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
2 \\
-1 \\
-1 \\
4
\end{bmatrix}, \begin{bmatrix}
-4 \\
5 \\
-10 \\
-6
\end{bmatrix}, \begin{bmatrix}
3 \\
-2 \\
10 \\
-1
\end{bmatrix}
\]

are linearly independent.
4. Find the value(s) of $c$ for which the matrix

$$
\begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & -1 & 2 & -1 \\
1 & -1 & c & -1 \\
-1 & 1 & 0 & 2
\end{bmatrix}
$$

is not invertible.
5. Solve the system

\[
\begin{align*}
2x_1 - x_2 + x_3 &= -5 \\
x_1 - x_3 &= 2 \\
-x_1 + 3x_2 + 2x_3 &= 1
\end{align*}
\]

using Cramer’s rule.
6. Determine whether the following sets are subspaces of the appropriate $\mathbb{R}^n$.

\( (a) \quad V_a \equiv \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1^3 + 335x_2^7 + 70x_3 + \pi x_4^5 = 11 \right\} ; \)

\( (b) \quad V_b \equiv \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^3 : u_1u_2 = 2u_3^3 \right\} ; \)

\( (c) \quad V_c \equiv \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_1 + u_2 \leq 1, u_3 + u_4^3 \leq 2 \right\} ; \)

\( (d) \quad V_d \equiv \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} ; \)

\( (e) \quad V_e \equiv \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 5 \end{vmatrix} = 0 \right\} ; \)

\( (f) \quad V_f \equiv \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \mathbb{R}^4 : 2x + y + z - t = 0 \right\} . \)
7. Consider the matrix

\[
A = \begin{bmatrix}
1 & -2 & 0 & 2 \\
-1 & 2 & 1 & -3 \\
2 & -4 & 3 & 1
\end{bmatrix}.
\]

(a) Find a basis for the column space of \( A \). What is the dimension of the column space of \( A \)?

(b) Find a basis for the null space of \( A \). What is the dimension of the null space of \( A \)?
8. Is the set
\[
\left\{ \begin{bmatrix}
3r - 5s \\
3s \\
0 \\
-r + 11s
\end{bmatrix} \in \mathbb{R}^4 : r, s \in \mathbb{R} \right\}
\]
a subspace of \( \mathbb{R}^4 \)? If so, then find a basis for this subspace and determine the dimension of this subspace.
9. Let $V$ and $W$ be nonzero subspaces of $\mathbb{R}^n$ such that each vector $u$ in $\mathbb{R}^n$ can be uniquely expressed in the form $u = v + w$ for some $v$ in $V$ and some $w$ in $W$.

(a) Prove that $0$ is the only vector in both $V$ and $W$.
(b) Prove that $\dim V + \dim W = n$. 
10. Consider the matrix

\[
A = \begin{bmatrix}
-7 & 5 & 4 \\
0 & -3 & 0 \\
-8 & 9 & 5
\end{bmatrix}.
\]

(a) Determine the eigenvalues of \( A \) and their multiplicities.
(b) Find a basis for each eigenspace corresponding to each of the eigenvalues of \( A \).
(c) Is \( A \) diagonalizable?