Some of the problems on the final exam will be chosen from the following list [up to a change in notation]. The actual exam problems may have multiple parts and be a combination of some of the questions below.

1. State and prove the triangle inequality. [p12]

2. Prove that a subset $D$ of $\mathbb{C}$ is open if and only if it contains no point of its boundary. [p25]

3. Assume that $\{z_n\}$ and $\{w_n\}$ are convergent sequences of complex numbers. Let $\lambda \in \mathbb{C}$. Prove that $\{z_n + \lambda w_n\}$ is convergent and that
   \[
   \lim_{n \to \infty} (z_n + \lambda w_n) = \left( \lim_{n \to \infty} z_n \right) + \lambda \left( \lim_{n \to \infty} w_n \right).
   \]
   Prove that $\{z_n w_n\}$ is convergent and that
   \[
   \lim_{n \to \infty} z_n w_n = \left( \lim_{n \to \infty} z_n \right) \left( \lim_{n \to \infty} w_n \right).
   \]
   [p34 and your lecture notes]

4. Assume that $\{z_n\}$ and $\{w_n\}$ are convergent sequences of complex numbers such that
   \[
   \lim_{n \to \infty} w_n \neq 0.
   \]
   Prove that the sequence $\left\{ \frac{z_n}{w_n} \right\}$ is convergent and that
   \[
   \lim_{n \to \infty} \frac{z_n}{w_n} = \frac{\lim_{n \to \infty} z_n}{\lim_{n \to \infty} w_n}.
   \]
   [lecture notes]

5. Let $g$ be a complex-valued continuous function on $[a, b]$. Prove that
   \[
   \left| \int_a^b g(t) \, dt \right| \leq \int_a^b |g(t)| \, dt.
   \]
   Hint: There exists $r \geq 0$ and $\theta \in \mathbb{R}$ such that $\int_a^b g(t) \, dt = re^{i\theta}$. Use the fact that $\text{Re } z \leq |z|$. [p60-61]

6. Assume that $\gamma$ is a piecewise $C^1$ curve and that $u$ is a continuous function on the range of $\gamma$. Obtain an upper bound for $\left| \int_{\gamma} u(z) \, dz \right|$ which holds without any additional assumptions on $\gamma$ and $u$. [p61-62 and lecture notes].

7. Let $\Omega$ be a domain whose boundary $\Gamma$ consists of a a finite number of disjoint, piecewise smooth simple closed curves. Assume that $f$ is a real-valued harmonic function on an open set which contains $\Omega$ and its boundary. Prove that $f = 0$ on $\Gamma$ if and only if $f = 0$ on $\Omega \cup \Gamma$.
   Hint: You may want to consider
   \[
   v = f \frac{\partial f}{\partial x} \quad \text{and} \quad u = -f \frac{\partial f}{\partial y}.
   \]
   [p73]

\[^1\text{A piecewise } C^1 \text{ curve is called piecewise smooth in the textbook.}\]
8. Prove that a \( C \)-differentiable function on an open set satisfies the Cauchy-Riemann equations. [p80]

9. Let \( f : U \rightarrow \mathbb{C} \) be a \( C \)-differentiable function. Prove that \( \text{Re} f \) is harmonic. You may use, without proving it, the fact that any \( C \)-differentiable function has complex derivatives of all orders. [p80-81]

10. Suppose that \( f = u + iv \) is \( C \)-differentiable on a domain \( D \), where \( u \) and \( v \) are real-valued. If either \( u \) is constant on \( D \) or \( u^2 + v^2 \) is constant on \( D \), then \( f \) is constant on \( D \). 
   \text{Hint: For the second part, you may begin by proving the claim for the case when } |f| = 1 \text{ or you could compute the first order partial derivatives of } u^2 + v^2. [p82]

11. Suppose that \( f = u + iv \), where \( u \) and \( v \) are real-valued. Assume that \( u \), \( v \), and their first order partial derivatives are continuous in an open disc centered at \( z_0 \). Prove that if \( u \) and \( v \) satisfy the Cauchy-Riemann equations at \( z_0 \) then \( f \) is \( C \)-differentiable at \( z_0 \) and obtain a formula for \( f'(z_0) \) in terms of first order partial derivatives of \( u \) and \( v \). [p83]

12. Suppose there is some \( z_1 \neq z_0 \) such that \( \sum a_n(z_1 - z_0)^n \) converges. Prove that for each \( z \) with \( |z - z_0| < |z_1 - z_0| \), the series \( \sum a_n(z - z_0)^n \) is absolutely convergent. [p93]

13. Assume that \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) has a positive or infinite radius of convergence \( R \). Prove that within the disc \( |z - z_0| < R \), \( f \) is infinitely \( C \)-differentiable. Obtain a general formula for the \( k \)-th derivative of \( f \) and for \( a_n \). [p97-99]

14. State and prove Cauchy’s theorem. For the proof you may assume that \( f \) is of class \( C^1 \). [p107]

15. Let \( f \) be a \( C \)-differentiable function on a simply-connected domain \( D \) and let \( \gamma \) be a piecewise \( C^1 \) closed curve in \( D \). Prove that 
   \[
   \int_{\gamma} f(z) \, dz = 0.
   \] [p110]

16. Let \( u \) be a real-valued harmonic function on a disc \( \{z : |z - z_0| < r\} \). Prove that there exists a \( C \)-differentiable function on this disc whose real part is \( u \). [p246]

17. Prove that if \( f \) is \( C \)-differentiable in a simply-connected domain \( D \), then there exists a \( C \)-differentiable function \( F \) on \( D \) with \( F' = f \) on \( D \). [p109-110]

18. State and prove Cauchy’s Integral Formula. 
   [p111 or your lecture notes]
   \text{Note: The textbook is quoting Example 10 in Section 6, Chapter 1 for the last part of the proof. The content of that example should be part of your proof.}

19. Let \( D \) be an open connected subset of \( \mathbb{C} \) and let \( f : D \rightarrow \mathbb{C} \) be a continuous function.
   Assume that 
   \[
   \int_{\gamma} f(z) \, dz = 0
   \]
for every triangle $\gamma$ that lies, together with its interior, in $D$.

Prove that $f$ is $C$-differentiable on $D$.

*Hint:* Fix an arbitrary point $z_0 \in D$ and choose an open disc centered at $z_0$ which is included in $D$. Prove that $f$ has a complex antiderivative on this disc.

[p129-130]

20. State and prove Liouville’s theorem regarding the bounded $C$-differentiable functions on the entire plane $\mathbb{C}$.

*Hint:* Consider the function $g(z) \equiv \frac{F(z) - F(0)}{z}$ where $F$ is $C$-differentiable on $\mathbb{C}$. Show that the function $g$ can be extended to a $C$-differentiable function $\tilde{g}$ on $\mathbb{C}$ and apply Cauchy’s Integral formula for $\tilde{g}$ and a circle centered at the origin of “big enough radius”. Show that $\tilde{g} = 0$.

[p130,131 and your lectures notes]

21. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1$. Let $f: D \rightarrow \mathbb{C}$ be a $C$-differentiable function on the open set $D$, $D \subseteq \mathbb{C}$. Assume that $\gamma([a, b]) \subseteq D$.

Prove that

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t) \quad \forall t \in [a, b].$$

[lecture notes or Lemma 1 within the solution to Problem H on Homework 4[Due Feb20]]

22. Assume that $\gamma: [a, b] \rightarrow \mathbb{C}$ is a piecewise $C^1$ curve. Let $f: D \rightarrow \mathbb{C}$ be a $C$-differentiable function on the open set $D$, $D \subseteq \mathbb{C}$. Assume that $\gamma([a, b]) \subseteq D$.

Prove that

$$\int_{\gamma} f'(z)dz = f(\text{endpoint}) - f(\text{initial point}),$$

where

initial point $\equiv \gamma(a)$ and endpoint $\equiv \gamma(b)$.

[see your lecture notes for the $C^1$ case and below for the general case].

There exists a partition

$$a = t_0 < t_1 < \ldots < t_m = b$$

of $[a, b]$ such that

$$\gamma_j \equiv \gamma\big|_{[t_j, t_{j+1}]}$$

is of class $C^1$ for all $j \in \{0, 1, \ldots, m - 1\}$. Using the $C^1$ case of the statement,

$$\int_{\gamma} f(z)dz = \sum_{j=0}^{m-1} \int_{\gamma_j} f(z)dz = \sum_{j=0}^{m-1} [f(\gamma(t_{j+1})) - f(\gamma(t_j))] = f(\gamma(b)) - f(\gamma(a)).$$
23. Prove the Cauchy estimates:

\[ |f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)| \]

whenever \( f \) is \( \mathbb{C} \)-differentiable on a domain containing \( \mathcal{D}_r(z_0) \).

Use the Cauchy estimates for \( n=1 \) to prove Liouville’s theorem by showing that the derivative of a bounded entire function is identically zero.

[see Exercises 18,19 on p133 and p123]

24. (a) Assume that \( f : D \rightarrow \mathbb{C} \) is a \( \mathbb{C} \)-differentiable function on an open connected subset \( D \) of \( \mathbb{C} \). Assume that \( z_0 \in D \) and \( f^{(k)}(z_0) = 0 \) for all \( k \in \mathbb{Z}_{\geq 0} \). Prove that \( f = 0 \) on \( D \).

(b) Assume that \( f : D \rightarrow \mathbb{C} \) is a \( \mathbb{C} \)-differentiable nonconstant function on an open connected subset \( D \) of \( \mathbb{C} \). Let \( Z(f) \equiv \{ z \in D : f(z) = 0 \} \). Prove that \( Z(f) \) is discrete [i.e. all points of \( Z(f) \) are isolated points.]

25. Prove the fundamental theorem of algebra using complex analysis.

26. Find all self-biholomorphisms of \( \mathbb{C} \).

Note: A self-biholomorphism of \( \mathbb{C} \) is a holomorphic [i.e. \( \mathbb{C} \)-differentiable] map \( f : \mathbb{C} \rightarrow \mathbb{C} \) such that \( f \) is invertible and \( f^{-1} \) is holomorphic.

Hint: Use a topological argument [involving compactness and the continuity of \( f^{-1} \)] to prove that \( \lim_{z \to \infty} f(z) = \infty \). What kind of singularity does \( f \) have at \( \infty \) and what does that say in terms of its Laurent expansion around \( \infty \)?

27. Find all self-biholomorphisms of \( \mathbb{C}\cup \{\infty\} \).

Note: Feel free to use the characterization of self-biholomorphisms of \( \mathbb{C} \) without proving it.

Hint: If \( f \) is a self-biholomorphism of \( \mathbb{C}\cup \{\infty\} \) such that \( f(\infty) = \infty \), then you are done. Otherwise, reduce the problem to this case by composing \( f \) with a convenient self-biholomorphism [or two] of \( \hat{\mathbb{C}} \).

28. Prove Schwarz’s lemma and then find all self-biholomorphisms of the open unit disc \( D_1(0) \).

Hint:

**Lemma 1** (Schwarz’s lemma). Let \( f \) be \( \mathbb{C} \)-differentiable on \( D_1(0) \). Assume that

\[ |f(z)| \leq 1 \quad \forall z \quad \text{and} \quad f(0) = 0. \]

Then \( |f(z)| \leq |z| \) and \( |f'(0)| \leq 1 \).

If either \( |f(z)| = |z| \) for some \( z \neq 0 \) or if \( |f'(0)| = 1 \), then \( f(z) = az \) for some \( a \in \mathbb{C} \) such that \( |a| = 1 \).

For the proof of Schwarz’s lemma:

Show that \( \frac{f(z)}{z} \) has a removable singularity at 0; what is the value at 0 of its holomorphic extension to \( D_1(0) \)? Use the maximum modulus principle to obtain an upper bound of the absolute value of this function on a disc \( \bar{D}_t(0) \) where \( 0 < t < 1 \). Think of the equality cases also from the perspective of the maximum modulus principle.

Then proceed following the hints below.
• Prove that if \( f : D_1(0) \to D_1(0) \) is a biholomorphism such that \( f(0) = 0 \), then \( f(z) = az \) for some \( a \in \mathbb{C} \) with \( |a| = 1 \). [use Schwarz’s lemma for \( f' \) and \( (f^{-1})' \) and the chain rule]. Prove a converse of this statement as well.

• For \( |a| < 1 \), let
  \[
  \phi_a(z) \equiv \frac{z - a}{1 - az}.
  \]
  Prove that \( \phi_a \) is a self-biholomorphism of \( D_1(0) \).

• If \( f \) is a self-biholomorphism of \( D_1(0) \) with \( f(0) = 0 \), then you are done. Otherwise, compose it with a convenient self-biholomorphism of \( D_1(0) \) and use the previous case.

29. State the Riemann mapping theorem.

Solution. Let \( D \subseteq \mathbb{C} \) be a simply-connected domain such that \( D \neq \mathbb{C} \). Then, \( D \) is biholomorphic to the open unit disc.

30. What are all simply-connected domains in \( \mathbb{C} \) up to biholomorphic equivalence?

31. What are all simply-connected domains in \( \mathbb{C} \) up to topological equivalence [i.e. homeomorphism]?

32. Define \( e^z \). Prove that \( F(z) \equiv e^z \) is entire and derive a formula for \( F'(z) \). What is the power series expansion of \( F \) about the origin? Where is it valid? [Your answer should be the “biggest” open set possible.]

Let \( y_0 \in \mathbb{R} \). Let
  \[
  A \equiv \{ z \in \mathbb{C} : \text{Im}z \in [y_0, y_0 + 2\pi) \}.
  \]
  Prove that \( F : A \to \mathbb{C}^* \) is bijective.
  Where is the function \( \log : \mathbb{C}^* \to A \) continuous/\( \mathbb{C} \)-differentiable?

At all points where \( \log : \mathbb{C}^* \to A \) is \( \mathbb{C} \)-differentiable, find a formula for its \( \mathbb{C} \)-derivative.

Consider the function \( \text{Log} : \mathbb{C}^* - \{ x \in \mathbb{R} : x < 0 \} \to \{ z \in \mathbb{C} : \text{Im}z \in (-\pi, \pi) \} \). What is the power series expansion of \( \text{Log}(1 - z) \) about \( 0 \)? Where is it valid? [Your answer should be a an open disc of maximum possible radius]

33. Derive power series expansions for \( \sin \) and \( \cos \) about \( 0 \). Where are they valid? [Your answer should be the “biggest” open set possible.]